Poset structure and enumerative results for a class of binary matrices equipped with a generalization of the Bruhat order

Alessandro Conflitti
(joint work with C.M. da Fonseca and Ricardo Mamede)

CMUC, Centre for Mathematics, University of Coimbra,
Apartado 3008, 3001-454 Coimbra, Portugal
E–mail: **conflitt@mat.uc.pt**
Web page: **http://www.mat.uc.pt/∼conflitt/**

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Outline

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Notation and background

$m, n \in \mathbb{N} \setminus \{0\}$
$R = (r_1, \ldots, r_m)$, $S = (s_1, \ldots, s_n)$ positive integral vectors.

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Definition

$$\mathscr{A}(R, S) := \{A = (a_{i,j}) \in M_{m,n}(\{0,1\}) \text{ s.t.}$$
$$\sum_{j=1}^{n} a_{\ell,j} = r_\ell, \quad \sum_{i=1}^{m} a_{i,t} = r_t,$$
$$\text{for all } 1 \leq \ell \leq m, 1 \leq t \leq n\}$$

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Question

For which R, S is $\mathscr{A}(R,S) \neq \emptyset$?

Answer

Gale-Ryser theorem!

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Interesting instance:
$m = n$, $k \in \mathbb{N} \setminus \{0\}$ such that $r_i = s_i = k$ for all $1 \le i \le n$.

In this case, write $\mathscr{A}(n, k)$ instead of $\mathscr{A}(R, S)$.

Generalization of Bruhat Order

Definition (Brualdi and Hwang)

Let R, S be such that $\mathscr{A}(R,S) \neq \emptyset$, and $A = (a_{i,j}) \in \mathscr{A}(R,S)$. $\Sigma_A = (\sigma_{ij}(A)) \in M_{m,n}(\{0,1\})$ such that

$$\sigma_{i,j}(A) := \sum_{\ell=1}^{i} \sum_{t=1}^{j} a_{\ell,t}, \quad 1 \leq i \leq m, 1 \leq j \leq n$$

If $A_1, A_2 \in \mathscr{A}(R,S)$, <u>define</u> $A_1 \preccurlyeq A_2$ if and only if $\Sigma_{A_1} \geqslant \Sigma_{A_2}$ in the entrywise order, i.e., $\sigma_{i,j}(A_1) \geq \sigma_{i,j}(A_2)$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Remark

$\mathscr{A}(\mathrm{n},1) \simeq \mathrm{S}_{\mathrm{n}}$, since it is the set of permutation matrices, and here $\preccurlyeq$ is nothing but the well known Bruhat order.

Examples

$$A=\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Examples

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad \Sigma_A = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 & 4 \\ 2 & 4 & 5 & 6 & 6 \\ 2 & 4 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 \end{pmatrix}$$

Examples

$$A=\begin{pmatrix}1&1&0&0&0\\1&1&0&0&0\\0&0&1&1&0\\0&0&1&0&1\\0&0&0&1&1\end{pmatrix}\qquad \Sigma_A=\begin{pmatrix}1&2&2&2&2\\2&4&4&4&4\\2&4&5&6&6\\2&4&6&7&8\\2&4&6&8&10\end{pmatrix}$$

$$B=\begin{pmatrix}1&1&0&0&0\\1&0&1&0&0\\0&1&1&0&0\\0&0&0&1&1\\0&0&0&1&1\end{pmatrix}$$

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A and B are not comparable in $(\mathscr{A}(5,2), \preccurlyeq)$.

Theorem by Brualdi and Deaett

Theorem

Let $n \in \mathbb{N} \setminus \{0\}$ and $0 \leq k \leq n$.
$(\mathscr{A}(n, k), \preccurlyeq)$ admits a unique minimal element if and only if $k \in \{0, 1, n-1, n\}$ or $n = 2k$.

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The minimal matrix in $\mathscr{A}(2k, k)$ is

$$P_k = J_k \oplus J_k = \begin{pmatrix} J_k & O_k \\ O_k & J_k \end{pmatrix},$$

where J_k is the matrix of all 1's of order k and O_k is the zero matrix also of order k, and the maximal matrix is

$$Q_k = \begin{pmatrix} O_k & J_k \\ J_k & O_k \end{pmatrix}.$$

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therefore the most interesting case is $\mathscr{A}(2k, k)$, the set of binary square matrices with all rows and columns having as many zeros as ones.

$\#\mathscr{A}(2k, k)$ is the sequence A058527 in the The On-Line Encyclopedia of Integer Sequences and computing a closed formula for such sequence is an open problem which looks quite hard.

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1. In $(\mathscr{A}(2\mathrm{k},\mathrm{k}), \preccurlyeq)$ is the maximal length of a chain equal to $4\mathrm{k}^2$?
2. What is the largest size of an antichain in $(\mathscr{A}(2\mathrm{k},\mathrm{k}), \preccurlyeq)$?

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Theorem

For any integer $k \geq 2$, the maximal length of a chain in $(\mathscr{A}(2k, k), \preccurlyeq)$ equals k^4,

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Our proofs are constructive.

We prove that the length of a chain in $(\mathscr{A}(2k,k), \preccurlyeq)$ must be at most k^4, and we design an algorithm which, for any integer $k \geq 2$, explicitly generates a chain of length k^4 and an antichain of size $\left(\left\lfloor \frac{k}{2} \right\rfloor^4 + 1\right)^2$.

Example: maximal chain when k = 2

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

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Vague sketch of proof for the antichain algorithm

Definition

Call Chain our algorithm which generates a chain of maximal length n^4 between P_n and Q_n, for any integer $n \geqslant 2$, and Rev–Chain its reverse, viz. the algorithm which generates the same chain backwards from Q_n and P_n.

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Consider the easiest case: $k \equiv 0 \pmod 2$, and let A be the half–way matrix of both Chain and Rev–Chain, i.e. the matrix generated at step $\frac{k^4}{2}$ by both algorithm.

$$
A = \left(\begin{array}{cc|cc} J_{\frac{k}{2}} & J_{\frac{k}{2}} & O_{\frac{k}{2}} & O_{\frac{k}{2}} \\ O_{\frac{k}{2}} & O_{\frac{k}{2}} & J_{\frac{k}{2}} & J_{\frac{k}{2}} \\ \hline O_{\frac{k}{2}} & O_{\frac{k}{2}} & J_{\frac{k}{2}} & J_{\frac{k}{2}} \\ J_{\frac{k}{2}} & J_{\frac{k}{2}} & O_{\frac{k}{2}} & O_{\frac{k}{2}} \end{array}\right) = \left(\begin{array}{c|c|c} \begin{matrix} J_{\frac{k}{2}} \\ O_{\frac{k}{2}} \end{matrix} & P_{\frac{k}{2}} & \begin{matrix} O_{\frac{k}{2}} \\ J_{\frac{k}{2}} \end{matrix} \\ \hline \begin{matrix} O_{\frac{k}{2}} \\ J_{\frac{k}{2}} \end{matrix} & Q_{\frac{k}{2}} & \begin{matrix} J_{\frac{k}{2}} \\ O_{\frac{k}{2}} \end{matrix} \end{array}\right)
$$

$$
= \left(\begin{array}{c|c|c} \begin{matrix} J^{*}_{\frac{k}{2}} \\ O^{*}_{\frac{k}{2}} \end{matrix} & P^{\bullet}_{\frac{k}{2}} & \begin{matrix} O^{*}_{\frac{k}{2}} \\ J^{*}_{\frac{k}{2}} \end{matrix} \\ \hline \begin{matrix} O^{\dagger}_{\frac{k}{2}} \\ J^{\dagger}_{\frac{k}{2}} \end{matrix} & Q^{\odot}_{\frac{k}{2}} & \begin{matrix} J^{\dagger}_{\frac{k}{2}} \\ O^{\dagger}_{\frac{k}{2}} \end{matrix} \end{array}\right)
$$

▸ To full antichain

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This process generates $(\frac{k}{2})^4 + 1$ elements incomparable, as well. This is not at all trivial, and requires a careful proof!

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Analogously, apply simultaneously Chain and Rev–Chain algorithms to the submatrices $*$ and $\dagger$, denoting this operation as lateral–antichain algorithm.

Go to matrix A

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This process generates $(\frac{k}{2})^4 + 1$ elements incomparable, as well. Again, this is not at all trivial, and requires a careful proof!

In fact, it is possible to apply independently both central–antichain and lateral–antichain algorithms, obtaining an antichain of size

$$\left(\left(\frac{k}{2}\right)^4 + 1\right)^2 .$$

Once more, the proof is quite lengthy and sophisticated!

Go to matrix A

References

References

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