Poset structure and enumerative results for a class of binary matrices equipped with a generalization of the Bruhat order

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Outline



- Preliminaries
- Bruhat Order for Binary Matrices

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1 Overview

- Preliminaries
- Bruhat Order for Binary Matrices
- 2 Known Results and Problems
 - Statements and Remarks
 - Open Problems and Questions

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3 Our Theorems

- Statements and Comments
- Instance of Chain of Maximal Length

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- 4 Extremely Vague Sketch of Proof

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Overview ts and Problems

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Notation and background

$$\label{eq:rescaled} \begin{split} m,n \in \mathbb{N} \setminus \{0\} \\ R = (r_1,\ldots,r_m), \, S = (s_1,\ldots,s_n) \text{ positive integral vectors.} \end{split}$$

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Notation and background

$$\label{eq:mn} \begin{split} m,n \in \mathbb{N} \setminus \{0\} \\ R = (r_1,\ldots,r_m), \, S = (s_1,\ldots,s_n) \text{ positive integral vectors.} \end{split}$$

Definition

$$\begin{split} \mathscr{A}(R,S) &:= \{A = (a_{i,j}) \in M_{m,n}\left(\{0,1\}\right) \ s.t. \\ &\sum_{j=1}^n a_{\ell,j} = r_\ell, \quad \sum_{i=1}^m a_{i,t} = r_t, \\ &\text{for all } 1 \leq \ell \leq m, 1 \leq t \leq n \} \end{split}$$

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Question

For which R, S is $\mathscr{A}(R, S) \neq \emptyset$?

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Answer

Gale-Ryser theorem!

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Answer

Gale-Ryser theorem!

Interesting instance:

 $m = n, k \in \mathbb{N} \setminus \{0\}$ such that $r_i = s_i = k$ for all $1 \le i \le n$.

In this case, write $\mathscr{A}(n, k)$ instead of $\mathscr{A}(R, S)$.

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Generalization of Bruhat Order

Definition (Brualdi and Hwang)

Let R, S be such that $\mathscr{A}(R,S) \neq \emptyset$, and $A = (a_{i,j}) \in \mathscr{A}(R,S)$. $\Sigma_A = (\sigma_{ij}(A)) \in M_{m,n}(\{0,1\})$ such that

$$\sigma_{i,j}(A) := \sum_{\ell=1}^{i} \sum_{t=1}^{j} a_{\ell,t}, \quad 1 \leq i \leq m, 1 \leq j \leq n$$

If $A_1, A_2 \in \mathscr{A}(\mathbb{R}, \mathbb{S})$, <u>define</u> $A_1 \preccurlyeq A_2$ if and only if $\Sigma_{A_1} \geqslant \Sigma_{A_2}$ in the entrywise order, i.e., $\sigma_{i,j}(A_1) \ge \sigma_{i,j}(A_2)$, for all $1 \le i \le m$ and $1 \le j \le n$.

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Remark

 $\mathscr{A}(n,1) \simeq S_n$, since it is the set of permutation matrices, and here \preccurlyeq is nothing but the well known Bruhat order.

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Examples

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

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Examples

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad \Sigma_{A} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 & 4 \\ 2 & 4 & 5 & 6 & 6 \\ 2 & 4 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 \end{pmatrix}$$

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Examples

 $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad \Sigma_A = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 & 4 \\ 2 & 4 & 5 & 6 & 6 \\ 2 & 4 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$

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Examples

(1	1	0	0	$0 \rangle$		(1	2	2	2	2
	1	1	0	0	0		2	4	4	4	4
A=	0	0	1	1	0	$\Sigma_{\mathrm{A}} =$	2	4	5	6	6
	0	0	1	0	1		2	4	6	7	8
	0	0	0	1	1)		$\backslash 2$	4	6	8	10/
							•				
(1	1	0	0	$\dot{0}$		(1)	2	2	2	2
ĺ	1 1	$\begin{array}{c} 1 \\ 0 \end{array}$	$\begin{array}{c} 0 \\ 1 \end{array}$	$\begin{array}{c} 0 \\ 0 \end{array}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$		$\begin{pmatrix} 1\\ 2 \end{pmatrix}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{4}$	$\begin{pmatrix} 2 \\ 4 \end{pmatrix}$
B=	1 1 0	$egin{array}{c} 1 \\ 0 \\ 1 \end{array}$	$egin{array}{c} 0 \ 1 \ 1 \end{array}$	0 0 0	$\begin{pmatrix} 0\\0\\0\\0 \end{pmatrix}$	$\Sigma_{ m B} =$	$\begin{pmatrix} 1\\ 2\\ 2 \end{pmatrix}$	2 3 4	2 4 6	2 4 6	$\begin{pmatrix} 2\\ 4\\ 6 \end{pmatrix}$
B=	1 1 0 0	1 0 1 0	$0 \\ 1 \\ 1 \\ 0$	$0 \\ 0 \\ 0 \\ 1$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$	$\Sigma_{\mathrm{B}} =$	$\begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$	$2 \\ 3 \\ 4 \\ 4$	$2 \\ 4 \\ 6 \\ 6$	$2 \\ 4 \\ 6 \\ 7$	$ \begin{array}{c} 2\\ 4\\ 6\\ 8 \end{array} $

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Examples

 $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad \Sigma_{A} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 & 4 \\ 2 & 4 & 5 & 6 & 6 \\ 2 & 4 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad \Sigma_B = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 3 & 4 & 4 & 4 \\ 2 & 4 & 6 & 6 & 6 \\ 2 & 4 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 \end{pmatrix}$

A and B are not comparable in $(\mathscr{A}(5,2),\preccurlyeq)$.

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Theorem by Brualdi and Deaett

Theorem

Let $n \in \mathbb{N} \setminus \{0\}$ and $0 \le k \le n$. ($\mathscr{A}(n,k), \preccurlyeq$) admits a unique minimal element if and only if $k \in \{0, 1, n - 1, n\}$ or n = 2k.

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Theorem by Brualdi and Deaett

Theorem

Let $n \in \mathbb{N} \setminus \{0\}$ and $0 \le k \le n$. $(\mathscr{A}(n,k),\preccurlyeq)$ admits a unique minimal element if and only if $k \in \{0, 1, n - 1, n\}$ or n = 2k.

The minimal matrix in $\mathscr{A}(2k, k)$ is

$$P_k = J_k \oplus J_k = \begin{pmatrix} J_k & O_k \\ O_k & J_k \end{pmatrix},$$

where J_k is the matrix of all 1's of order k and O_k is the zero matrix also of order k, and the maximal matrix is

$$\mathbf{Q}_{\mathbf{k}} = \begin{pmatrix} \mathbf{O}_{\mathbf{k}} & \mathbf{J}_{\mathbf{k}} \\ \mathbf{J}_{\mathbf{k}} & \mathbf{O}_{\mathbf{k}} \end{pmatrix}$$

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Remarks

There are 5 cases: $k \in \{0, 1, n - 1, n\}$ or n = 2k, but

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• $\mathscr{A}(n,k) \simeq \mathscr{A}(n,n-k)$ (swap 0's and 1's)

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 $\bullet \ \mathscr{A}(n,k) \simeq \mathscr{A}(n,n-k) \quad ({\rm swap} \ 0{\rm 's \ and} \ 1{\rm 's})$

•
$$\mathscr{A}(n,0) = \{O_n\}$$

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$$\mathscr{A}(n,1) \simeq S_n$$

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therefore the most interesting case is $\mathscr{A}(2k, k)$, the set of binary square matrices with all rows and columns having as many zeros as ones.

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therefore the most interesting case is $\mathscr{A}(2k, k)$, the set of binary square matrices with all rows and columns having as many zeros as ones.

 $\#\mathscr{A}(2k, k)$ is the sequence A058527 in the The On-Line Encyclopedia of Integer Sequences and computing a closed formula for such sequence is an open problem which looks quite hard.

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Questions by Brualdi and Deaett

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Questions by Brualdi and Deaett

■ In $(\mathscr{A}(2k, k), \preccurlyeq)$ is the maximal length of a chain equal to $4k^2$?

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- In $(\mathscr{A}(2k,k), \preccurlyeq)$ is the maximal length of a chain equal to $4k^2$?
- ② What is the largest size of an antichain in $(\mathscr{A}(2k,k), \preccurlyeq)$?

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Our results

Theorem

For any integer $k \ge 2$, the maximal length of a chain in $(\mathscr{A}(2k,k),\preccurlyeq)$ equals k^4 ,

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Theorem

For any integer $k \ge 2$, the maximal length of a chain in $(\mathscr{A}(2k,k),\preccurlyeq)$ equals k^4 , and the largest size of an antichain is at least

$$\left(\left\lfloor\frac{\mathbf{k}}{2}\right\rfloor^4 + 1\right)^2.$$

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Our results

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For any integer $k \ge 2$, the maximal length of a chain in $(\mathscr{A}(2k,k),\preccurlyeq)$ equals k^4 , and the largest size of an antichain is at least

$$\left(\left\lfloor\frac{\mathbf{k}}{2}\right\rfloor^4+1\right)^2$$
.

Our proofs are constructive.

We prove that the length of a chain in $(\mathscr{A}(2k, k), \preccurlyeq)$ must be at most k^4 , and we design an algorithm which, for any integer $k \ge 2$, explicitly generates a chain of length k^4 and an antichain of size $\left(\left\lfloor \frac{k}{2} \right\rfloor^4 + 1\right)^2$.

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Example: maximal chain when k = 2

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

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Example: maximal chain when k = 2

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

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$$\mapsto \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

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Vague sketch of proof for the antichain algorithm

Definition

Call <u>Chain</u> our algorithm which generates a chain of maximal length n^4 between P_n and Q_n , for any integer $n \ge 2$, and <u>Rev–Chain</u> its reverse, viz. the algorithm which generates the same chain backwards from Q_n and P_n .

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Consider the easiest case: $k \equiv 0 \pmod{2}$, and let A be the half-way matrix of both Chain and Rev–Chain, i.e. the matrix generated at step $\frac{k^4}{2}$ by both algorithm.

$$\begin{split} \mathbf{A} &= \begin{pmatrix} \mathbf{J}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} & \mathbf{O}_{\frac{k}{2}} & \mathbf{O}_{\frac{k}{2}} \\ \mathbf{O}_{\frac{k}{2}} & \mathbf{O}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} \\ \mathbf{O}_{\frac{k}{2}} & \mathbf{O}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} \\ \mathbf{J}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} & \mathbf{O}_{\frac{k}{2}} & \mathbf{O}_{\frac{k}{2}} \\ \mathbf{J}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} & \mathbf{O}_{\frac{k}{2}} & \mathbf{O}_{\frac{k}{2}} \\ \mathbf{J}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} & \mathbf{O}_{\frac{k}{2}} & \mathbf{O}_{\frac{k}{2}} \\ \mathbf{J}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} & \mathbf{O}_{\frac{k}{2}} \\ \mathbf{O}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} & \mathbf{O}_{\frac{k}{2}} \\ \mathbf{J}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} \\ \mathbf{O}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} \\ \mathbf{J}_{\frac{k}{2}}^{\dagger} & \mathbf{Q}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} \\ \mathbf{J}_{\frac{k}{2}}^{\dagger} & \mathbf{J}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} \\ \mathbf{J}_{\frac{k}{2}}^{\dagger} & \mathbf{J}_{\frac{k}{2}} & \mathbf{J}_{\frac{k}{2}} \\ \mathbf{J}_{\frac{k}{2}}^{\dagger} & \mathbf{J}_{\frac{k}{2}} \\ \mathbf{J}_{\frac{k}{2$$

▸ To full antichain

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Apply simultaneously Chain and Rev–Chain algorithms to \bullet and $^{\odot}$, and denote this operation as central–antichain algorithm.

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Analogously, apply simultaneously Chain and Rev–Chain algorithms to the submatrices * and † , denoting this operation as lateral–antichain algorithm.

▶ Go to matrix A

Apply simultaneously Chain and Rev–Chain algorithms to • and $^{\odot}$, and denote this operation as central–antichain algorithm. This process generates $(\frac{k}{2})^4 + 1$ elements incomparable, as well. This is not at all trivial, and requires a careful proof!

Analogously, apply simultaneously Chain and Rev–Chain algorithms to the submatrices * and † , denoting this operation as lateral–antichain algorithm.

Go to matrix A

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This process generates $(\frac{k}{2})^4 + 1$ elements incomparable, as well. Again, this is not at all trivial, and requires a careful proof!

In fact, it is possible to apply independently both central–antichain and lateral–antichain algorithms, obtaining an antichain of size

$$\left(\left(\frac{\mathrm{k}}{2}\right)^4+1\right)^2$$

Once more, the proof is quite lengthy and sophisticated!

▶ Go to matrix A

References

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