

# Polynomial analogues of Ramanujan congruences for Han's hooklength formula

William J. Keith  
CELC, University of Lisbon

Email: [william.keith@gmail.com](mailto:william.keith@gmail.com)

Detailed arXiv preprint: [1109.1236](https://arxiv.org/abs/1109.1236)

# Context

## Partition congruences

The generating function for the number of partitions of  $n$  is

$$P(q) = \prod_{k=1}^{\infty} (1 - q^k)^{-1} = \sum_{n \geq 0} p(n)q^n .$$

Ramanujan first observed that its coefficients satisfy congruences in arithmetic progression, such as  $p(5k + 4) \equiv 0 \pmod{5}$  and  $p(11k + 6) \equiv 0 \pmod{11}$ , among others. This work was extended to an infinite family of congruences through work of many mathematicians, including Ono, Bringmann, and Mahlburg, based on the Andrews-Garvan-Dyson crank.

# Context

## Partition power congruences

Powers of the partition function can be interpreted, when positive, as *multipartitions*, vectors of partitions, and when negative as shifted powers of the eta function.

$$P_r(q) := \prod_{k=1}^{\infty} (1 - q^k)^{-r} = \sum_{n \geq 0} p_r(n) q^n .$$

These functions also satisfy infinite families of congruences, and were studied by Newman, Andrews, Atkin, and Serre, among others, the methods of the latter several being the theory of modular forms. Ramanujan himself did some work with these using elementary means, explicated in a recent paper of Berndt, Gugg and Kim.

# Context

## The hooklength formula

Discovered independently (and by very different means) first by Nekrasov and Okounkov, and then by Guo-Niu Han, the hooklength formula beautifully relates these powers to a sum over the set of partitions  $\mathcal{P}$  and the hooklengths  $h_{ij}$  in those partitions.

$$\prod_{k=1}^{\infty} (1 - q^k)^{b-1} = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h_{ij} \in \lambda} \left( 1 - \frac{b}{h_{ij}^2} \right) =: \sum_{n \geq 0} \frac{q^n}{n!} p_n(b)$$

The coefficient of any  $q^n$  is clearly a polynomial in the  $b$ , and if we extract a factor of  $n!$  the coefficient  $p_n(b)$  is integral. (Han shows this, or an integral recurrence can be derived with Wilf's  $q \frac{\partial}{\partial q} \log$  technique.)

## Congruence analogues

The existence of families of congruences for  $P_r(q)$  suggests that the  $p_n(b)$  must obey some regularities. As it turns out, there seem to be more than we might have ever expected.

The  $p_n(b)$  have numerous symmetries for their coefficients in arithmetic progressions mod primes, which may be regarded as analogous to congruence results for their individual evaluations (e.g.,  $b = 0$  being the partition function).

# Congruence analogues

## Examples

For instance:  $p_4(b) = 120 - 218b + 119b^2 - 22b^3 + b^4$ , which reduced mod 5 gives  $p_4(b) \equiv 0 + 2b + 4b^2 + 3b^3 + b^4$ . The case  $p_9(b)$  has coefficients that reduce to:

$$(0, 0, 2, 4, 3, 1, 3, 1, 2, 4).$$

For the arithmetic progression  $5k + 4$ , we have the following congruence analogue:

# Congruence analogues

## Theorem

The  $p_n(b)$  defined by  $\prod_{k=1}^{\infty} (1 - q^k)^{b-1} = \sum_{n=0}^{\infty} \frac{q^n}{n!} p_n(b)$ , when  $n = 5k + 4$ , have integer coefficients with the symmetries:

- The nonzero residues mod 5 equally populate the residue classes 1, 2, 3, and 4.
- These appear in groups of four as a rotation of (2, 4, 3, 1).
- The  $k + 1$  terms of lowest degree are all 0 mod 5. (There may be others.)

The second and third clauses of the theorem tell us that we can identify the residue classes of the coefficients of  $p_{5k+4}(b)$  by taking just every fourth one after removing  $k + 1$  zeros at the front. If we do that, we get this array:





# Hooklength congruences

Almost anything you could conjecture about the apparent symmetry in this triangle is quite possibly true!

It turns out that this is Pascal's triangle, multiplied by 2 and an alternating sign, reduced mod 5, and proving this will give us our theorem.

## Proof sketch for main theorem

In order to get to newer work and open question, we'll simply outline the proof strategy, which will introduce some useful ideas.

We begin by expanding the product out using the generalized binomial theorem. Recall that for any value  $x$ , including the indeterminate in  $\mathbb{C}[x]$ , we can define the generalized binomial coefficient with a whole number  $k$  as 
$$\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}.$$

The binomial theorem for indeterminate powers, plus some algebra, gets us as far as:

## Proof sketch for main theorem

$$\prod_{j=0}^{\infty} (1 - q^j)^{b-1} = \sum_{n=0}^{\infty} \frac{q^n}{n!} \sum_t b^t \cdot \overbrace{(-1)^t n! \sum_{\substack{\lambda \vdash n \\ \lambda = 1^{e_1} 2^{e_2} \dots}} \left( \sum \frac{1}{s_1 \dots s_t} \right)}$$

where the last sum runs over unordered  $t$ -tuples of distinct elements chosen from the multiset  $\{1, 2, \dots, e_1, 1, 2, \dots, e_2, \dots\}$ . (We regard different instances of 1, etc., as distinct.)

In finding the residue class of the overbraced number number mod 5, a key observation is that if the power of 5 that divides  $n!$  ( $= (5k + 4)!$ ) is not fully canceled by elements of the product  $s_1 \dots s_t$ , that term will be 0 mod 5. It is possible for this to occur *only* if  $e_1 \geq 5k$ , and among the  $s_i$  are  $5, 10, 15, \dots, 5k$  chosen from  $\{1, 2, \dots, e_1, (\dots)\}$ .

## Proof sketch for main theorem

So  $e_1$  must be "big." That is, no matter how big  $k$  gets, the only partitions  $\lambda$  that contribute to the sum mod 5 are:

$$1^{5k+4} \quad , \quad 1^{5k+2}2^1 \quad , \quad 1^{5k+1}3^1 \quad , \quad 1^{5k}4^1 \quad , \quad 1^{5k}2^2 \quad .$$

From the five multisets associated to these partitions we choose  $t$  elements in all ways to construct the coefficient of  $b^t$ .

By the previous argument,  $k$  of our choices are already "spoken for" if we wish to count the nonzero contributions:  $s_1 = 5$ ,  $s_2 = 10$ ,  $\dots$ ,  $s_k = 5k$ . If  $t = k + m$ , we have  $m$  possible choices remaining.

## Proof sketch for main theorem

It turns out (for details, see the preprint) that terms can be grouped with many choices occurring in groups that are multiples of size 5. We find that the coefficient on  $b^t$  in  $p_{5k+4}(b)$ ,  $t = k + m$ ,  $m > 0$ , is

$$(-1)^{m+1} \sum_{c=1}^4 a_c f_k(m-c)$$

where  $a_1 = 2$ ,  $a_2 = 1$ ,  $a_3 = 3$ , and  $a_4 = 4$ , the  $m - c$  records how many  $s_i$  come from the first  $k$  sets of  $\{1, 2, 3, 4\}$  in  $\{1, \dots, e_1, \dots\}$ , and the  $f_k$  are defined for any prime  $p$  by:

$$f_k(m) := \sum_{r_1 + \dots + r_{p-1} = m} \binom{k}{r_1} \dots \binom{k}{r_{p-1}} 1^{r_1} 2^{r_2} \dots (p-1)^{r_{p-1}}.$$

## Proof sketch for main theorem

The evaluation of this sum is easy in the arithmetic progressions that interest us:

### Lemma

For  $p$  a prime,  $f_k((p-1)s+c) :=$

$$\sum_{r_1+\dots+r_{p-1}=(p-1)s+c} \binom{k}{r_1} \dots \binom{k}{r_{p-1}} 1^{r_1} 2^{r_2} \dots (p-1)^{r_{p-1}}$$
$$\equiv \begin{cases} (-1)^s \binom{k}{s} \pmod{p} & c=0 \\ 0 \pmod{p} & \text{otherwise.} \end{cases}$$

The proof is simply to observe that the middle term is the coefficient of  $q^{(p-1)s+c}$  in the product

$$(1+q)^k (1+2q)^k \dots (1+(p-1)q)^k \equiv (1-q^{p-1})^k \pmod{p}.$$

## Proof sketch for main theorem

This proves the theorem. For  $p = 5$ , only every fourth underlying sum  $f_k(4s + 0)$  is nonzero, so every other term is  $(-1)^{m+1}a_c$  times this value. This is either  $(2, -1, 3, -4)$  or  $(-2, 1, -3, 4)$  times the underlying  $\binom{k}{s}$ . Thus, terms are equidistributed in the nonzero residue classes, in rotations of  $(2, 4, 3, 1)$ .

The symmetries of binomial coefficients explain the triangle we showed earlier: this is Pascal's triangle, multiplied by 2 and an alternating sign, reduced mod 5.



## Brief tangent: an identity from the coefficients

$$(-1^t) [b^t] \sum_{\lambda \vdash n} \prod_{h_{ij} \in \lambda} \left( 1 - \frac{b}{h_{ij}^2} \right) = \sum_{\lambda = 1^{e_1} 2^{e_2} \dots} \left( \sum \frac{1}{s_1 \dots s_t} \right)$$

where the right hand side can be thought of as the sum of the  $t$ -th elementary symmetric functions in the elements of each multiset. These elements are a proper subset of the hooklengths themselves.

1							
4	2	1					
5	3	2					
10	8	7	4	3	2	1	

$$\lambda = 1^4 2^0 3^2 4^1 \dots$$

The red elements are the values that appear in the multiset.

The identity cannot be refined to the single-partition level; can it be refined otherwise?



## Other primes

Equidistribution does not happen for other primes  $p$  in the residue class  $p - 1$  (at least not up to  $p < 800$ ):

$$p_6(b) = 7920 - 18144b + 14674b^2 - 5205b^3 + 805b^4 - 51b^5 + b^6 \\ \equiv (3, 0, 2, 3, 0, 5, 1) \pmod{7} .$$

The main reason is that the  $a_c$  are not as neatly distributed, especially including the fact that  $a_0$  is not usually 0. However, observe mod 7:

$$p_{41}(b) \equiv (0, 0, 0, 0, 3, 0, 2, 3, 0, 5, 0, 0, 4, 6, 0, 3, 4, 0, 6, 2, 0, 1, \dots, 6)$$

## Other primes

$p_{41}(b) \equiv (0, 0, 0, 0, 3, 0, 2, 3, 0, 5, 0, 0, 4, 6, 0, 3, 4, 0, 6, 2, 0, 1, \dots, 6)$

We still get equidistribution in one of every  $p^2$  or  $p^3$  progressions, because the binomial coefficients themselves rotate through the residue classes mod  $p$ . In particular,

### Lemma

For  $p$  a prime,  $0 \leq s \leq pj + p - 2$ ,  $s = gp + h$ ,  $0 \leq h < p$ ,

$$(-1)^s \binom{pj + p - 2}{s} \equiv (h + 1) \left( (-1)^g \binom{j}{g} \right) \pmod{p} .$$

Thus the interiors of the intervals  $(p - 1)s + c$  rotate through the multiples  $h + 1 \pmod{p}$ , and the ends overlap but still rotate – usually properly, with one exception.

## Other primes

The sequence of reduced coefficients in  $p_{(pj+p-2)p+p-1}(b)$  are thus:

$$1 \leq c < p - 1: \quad \{(-1)^{c+1} a_c (-1)^s \binom{pj + p - 2}{s}\}$$

which are segments of length  $p$  that are either all 0s, or permutations of  $\{1, \dots, p - 1\}$  followed by a 0, and for the ends,

$$\{-a_0 (-1)^s \binom{pj + p - 2}{s} - a_{p-1} (-1)^{s-1} \binom{pj + p - 2}{s-1}\}$$

for  $0 \leq s \leq pj + p - 1$ ,  $s = gp + h$ ,  $0 \leq h < p$ , which reduces to

$$\{(-1)^x \binom{j}{x} (-a_0 + y(-a_0 - a_{p-1})) : 0 \leq x \leq j, 0 \leq y \leq p - 1\}.$$

## Other primes

$$\{(-1)^x \binom{j}{x} (-a_0 + y(-a_0 - a_{p-1})) : 0 \leq x \leq j, 0 \leq y \leq p-1\}$$

But  $a_0$  is the number of partitions of  $p-1$ , and  $a_{p-1} \equiv -1 \pmod{p}$ . As long as the number of partitions of  $p-1$  is  $\not\equiv 1 \pmod{p}$ , we have that  $\{(-1)^x \binom{j}{x} (-a_0 + y(-a_0 - a_{p-1}))\}$  runs over all residue classes mod  $p$  as  $y$  runs over  $0 \leq y \leq p-1$ .

Even if the number of partitions of  $p-1$  is  $\equiv 1 \pmod{p}$  (it happens for the first time at  $p=71$ ), we can apply the lemma again to  $(-1)^x \binom{j}{x}$  to see that the sequence still equally populates the residue classes when  $j \equiv -2 \pmod{p}$ , which is the arithmetic progression  $-p^2 - p - 1 \pmod{p^3}$ .

## Other primes

So we have an infinite family of equidistributions mod every prime  $p$ , either in a progression mod  $p^2$ , or at worst mod  $p^3$ !

### Theorem

*For  $p$  prime,  $j \geq 0$ , if the number of partitions of  $p - 1$  is not congruent to 1 mod  $p$ , the coefficients of  $p_{p^2k-p-1}(b)$  equinumerously populate the nonzero residue classes mod  $p$  for all  $j$ , and if it is, the populations are still equinumerous for  $p_{p^3k-p^2-p-1}(b)$ .*

## Recent work

I first presented this work at the INTEGERS Conference at the University of West Georgia, this September. A member of the audience suggested that I look at the roots of these polynomials, conjecturing that the only integer roots would be very small.

I'd like to talk briefly about this and some other observations and open questions. Most of this work is experimental and speculative at the moment, and any interest would be happily received.

# Recent work

## Factorizations

Let us start by factoring some of the  $p_n(b)$ :

$$p_1(b) = -(b - 1)$$

$$p_2(b) = (b - 1)(b - 4)$$

$$p_3(b) = -(b - 1)(b - 2)(b - 9)$$

$$p_4(b) = (b - 1)(b - 2)(b - 4)(b - 15)$$

$$p_5(b) = -(b - 1)(b - 4)(b - 7)(b^2 - 23b + 30)$$

$$p_9(b) = -(b - 1)(b - 2)(b - 4)(b - 5)(b - 15)(b - 27) \\ (b^3 - 63b^2 + 614b - 674)$$

$$p_{45}(b) = -(b - 1)(b - 2)(b - 9)(595 \dots 000 - \dots + b^{42})$$

# Recent work

## Factorizations

The conjectures that arise are:

### Conjecture

- All  $p_n(b)$  factor as a squarefree product of several  $b - j$  and one large irreducible factor.
- The only  $j$  for which  $b - j$  appear are 1, 2, 3, 4, 5, 7, 9, 11, 15, and 27.

The second clause is false:  $j = 16$  arises at  $n = 53$ . But it is not very false, as we'll see momentarily.

We can also ask where a given factor of  $b - j$  arises.



# Recent work

## Factorizations

To say that  $b - j$  is a factor of  $p_n(b)$  is to say that the coefficient of  $q^n$  in  $\prod(1 - q^k)^{j-1}$  is 0. Thus, we are asking about the appearance of zero coefficients in powers of the eta function.

$$\prod(1 - q^k)^0 = 1 \Rightarrow b - 1 | p_n(b) \quad \forall n > 0$$

$$\prod(1 - q^k)^1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2+n}{2}} \Rightarrow b - 2 | p_n(b) n \neq \frac{3m^2 \pm m}{2}$$

$$\prod(1 - q^k)^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{\frac{n^2+n}{2}} \Rightarrow b - 4 | p_n(b) n \neq \frac{m^2 + m}{2}$$

# Recent work

## Factorizations

The remaining cases  $j = 3, 5, 7, 9, 11, 15, 27$  are exactly those powers of the  $\eta$  function that Serre showed to be lacunary, i.e. the arithmetic density of their nonzero coefficients is zero. That means these  $b - j$  appear frequently and infinitely often as  $n$  gets large. Of course, this does not tell us precisely where these factors appear.

Ken Ono showed Serre's conjecture that all other (positive) powers are not lacunary, so in a sense any other simple factors will appear rarely. It might be the case that  $b - 16$  appears only a finite number of times, if there exists an  $N$  beyond which the coefficient of  $q^n$  in  $\prod(1 - q^k)^{15}$  is always nonzero.

# Recent work

## Factorizations

To show that some linear factor never appears would be to show that  $\prod(1 - q^k)^{j-1}$  has *no* nonzero coefficients, which is a much stronger question: for instance, Lehmer's conjecture on Ramanujan's  $\tau$ -function asserts that  $b - 25$  never appears.

Note that the two claims are equivalent! So if we could show that a given factor never appears by combinatorial analysis of the expansion on slide 11, we would have shown that all coefficients in the desired power were nonzero, a very pleasing result.

Wide open is the question of why it seems that, beyond the linear factors, only a single irreducible factor appears. Why should it be the case that there is not, say, a couple of quadratics and another large factor? Why should each factor appear only once?

# Recent work

## Factorizations

Another way of looking at these polynomials is to factor them mod  $q$  for  $q$  prime, i.e. reduce their coefficients mod  $q$  and factor them over the field  $\mathbb{Z}_q$ .

In this case, a most fascinating thing happens: the factorization mod  $q$  of any  $p_{nq+c}(b)$  is just the factorization of  $p_{(n-1)q+c}$  times  $b^q - b = b(b-1)\dots(b-(q-1))$ . So mod  $q$ , you only need factor  $p_0$  through  $p_{q-1}$  to know all factorizations.

# Recent work

## Factorizations

Factorizations mod 5:

$$p_1(b) \quad 4(4 + b)$$

$$p_2(b) \quad (1 + b)(4 + b)$$

$$p_3(b) \quad 4(1 + b)(3 + b)(4 + b)$$

$$p_4(b) \quad b(1 + b)(3 + b)(4 + b)$$

$$p_5(b) \quad 4b(1 + b)(2 + b)(3 + b)(4 + b)$$

$$p_{46}(b) \quad b^9(1 + b)^9(2 + b)^9(3 + b)^9(4 + b)^{10}$$

$$p_{47}(b) \quad 4b^9(1 + b)^{10}(2 + b)^9(3 + b)^9(4 + b)^{10}$$

$$p_{48}(b) \quad b^9(1 + b)^{10}(2 + b)^9(3 + b)^{10}(4 + b)^{10}$$

$$p_{49}(b) \quad 4b^{10}(1 + b)^{10}(2 + b)^9(3 + b)^{10}(4 + b)^{10}$$

$$p_{50}(b) \quad b^{10}(1 + b)^{10}(2 + b)^{10}(3 + b)^{10}(4 + b)^{10}$$

# Recent work

## Factorizations

Things are not quite as neat for other primes. For example, higher-degree factors may arise. Factorizations mod 3 and 7:

$p_1(b)$	$2(2 + b)$	$6(6 + b)$
$p_2(b)$	$(2 + b)^2$	$(3 + b)(6 + b)$
$p_3(b)$	$2b(1 + b)(2 + b)$	$6(5 + b)^2(6 + b)$
$p_4(b)$	$b(1 + b)(2 + b)^2$	$(3 + b)(5 + b)(6 + b)^2$
$p_5(b)$	$2b(1 + b)(2 + b)^3$	$6b(3 + b)(6 + b)(2 + 5b + b^2)$
$p_6(b)$	$b^2(1 + b)^2(2 + b)^2$	$(3 + b)(5 + b)(6 + b)(4 + 5b^2 + b^3)$
$p_7(b)$	$2b^2(1 + b)^2(2 + b)^3$	$6b(1 + b)(2 + b) \dots (6 + b)$

# Recent work

## Parities

The factorization mod 2 is as regular as you would expect: alternating  $b$  and  $(1 + b)$ , the only two possible factors.

$$p_1(b) \quad (1 + b)$$

$$p_2(b) \quad b(1 + b)$$

$$p_3(b) \quad b(1 + b)^2$$

$$p_4(b) \quad b^2(1 + b)^2$$

$$p_5(b) \quad b^2(1 + b)^3$$

$$p_6(b) \quad b^3(1 + b)^3$$

This brings us to the problem of the parity of the partition function, a traditional Very Hard Question.

# Recent work

## Parities

The coefficient of  $q^n$  in  $\prod \frac{1}{(1-q^k)^2}$  is  $\sum_{i=0}^n p(n-i)p(i)$ . This is also the evaluation of  $\frac{1}{n!}p_n(b)$  at  $b = -1$ . Note that the constant term of  $p_n(b)$  is exactly  $p(n)n!$ , and subtract this from both sides. We get

$$p(n) + \sum_{i=1}^{n-1} p(n-i)p(i) = \frac{1}{n!} \sum_{i=1}^n |[b^t] p_n(b)| .$$

If  $n$  is odd, the left-hand side has the parity of  $p(n)$ , and if even, it has the parity of  $p(n) + p(\frac{n}{2})^2$ . Could we use this expression and facts on the frequency of multiplicities in all partitions of  $n$  to be able to say something about the parity of  $p(n)$ ?



# Recent work

## Parities

This triangle gives the 2-valuation of each of the coefficients in the first few  $p_n(b)$  for  $n$  odd. The 0s are all predicted by the earlier theory, so our interest is in the nonzero terms.

```
n=1: { 0, 0}
n=3: { 1, 0, 2, 0}
n=5: { 3, 1, 0, 0, 0, 0}
n=7: { 4, 2, 3, 0, 1, 3, 1, 0}
n=9: { 8, 4, 2, 3, 0, 0, 1, 1, 0, 0}
n=11: {11, 5, 3, 2, 1, 0, 2, 0, 1, 0, 4, 0}
n=13: {10, 7, 6, 3, 2, 1, 0, 0, 0, 0, 0, 0, 0}
n=15: {15, 8, 8, 4, 4, 3, 2, 0, 1, 1, 1, 2, 1, 1, 1, 0}
n=17: {15, 11, 8, 7, 4, 4, 3, 5, 0, 0, 2, 2, 1, 1, 2, 2, 0, 0}
n=19: {17, 12, 9, 9, 5, 4, 4, 3, 1, 0, 2, 0, 2, 1, 6, 1, 1, 0, 2, 0}
n=21: {21, 14, 11, 9, 7, 5, 4, 6, 4, 1, 0, 0, 0, 0, 1, 1, 1, 3, 0, 0, 0, 0}
```

Inspection suggests that there are significant regularities that might be tamed. For instance, the first diagonal below the 0s reads 1,1,3,3,1,1,2,5,1,1,..., and terms going backward from there steadily (if not quite monotonically) increase.

## Other open questions

It seems like a bit of a "lucky accident" that the values  $a_c$  distributed so neatly for  $p = 5$  that we had equidistribution mod  $p$ . No other prime checked experimentally has equidistribution in the arithmetic progression  $p - 1 \pmod p$ . Does it ever happen again? If so, how can we efficiently find it? If not, how would you prove it?

An intermediate step that might be useful in those questions would be to develop a short recipe for calculating the  $a_c$ , instead of going through and working out groupings and remainders by hand. A formula with a number of terms linear in  $p$  would be nice.

## Other open questions

Is there a combinatorial object that can be neatly described which the coefficients themselves (or their absolute values) count?

Perhaps such a description would be useful in producing identities for the coefficients of  $p_n(b)$ , or for associated objects such as multipartitions.

## Other open questions

What can we say about progressions with composite moduli other than prime powers?

## Other open questions

Finally, regarding the asymptotics, a normalized plot of the logs of the absolute values of the coefficients appears to approach a very well-behaved curve, with the maximum staying extremely close to the beginning.

It seems clear that the coefficients of the  $p_n$  are alternating unimodular, and my current interest is in showing that they satisfy a sufficiently restrictive growth condition that  $(-1)^n p_n(b)$  can generate  $\{1, b, b^2, b^3, \dots\}$  with positive integer coefficients.

If any questions about these polynomials interest *you*, I'd be happy to take a look at them!

## References

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