Monochromatic Clique Decompositions of Graphs

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Joint work with Henry Liu, UNL.

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- Dor &Tarsi ['97]: NP-hard if H has a component with ≥ 3 edges

The H-decomposition problem

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- Turán number: $t_p(n) = e(T_p(n))$

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- K_{p+1}-free
- Maximum number of edges

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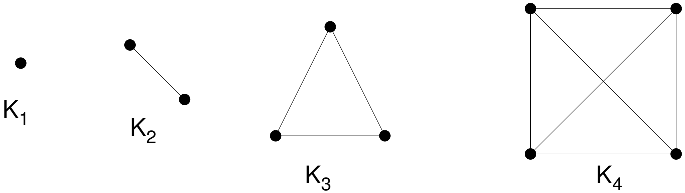
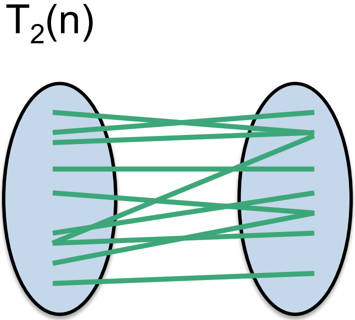
K_p = Complete graph on p vertices = Clique



Turán Graph $T_2(n)$



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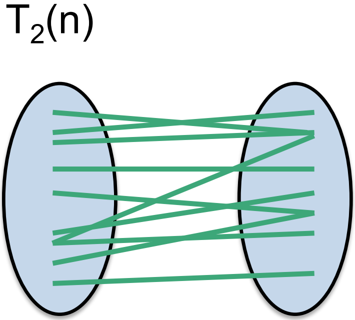
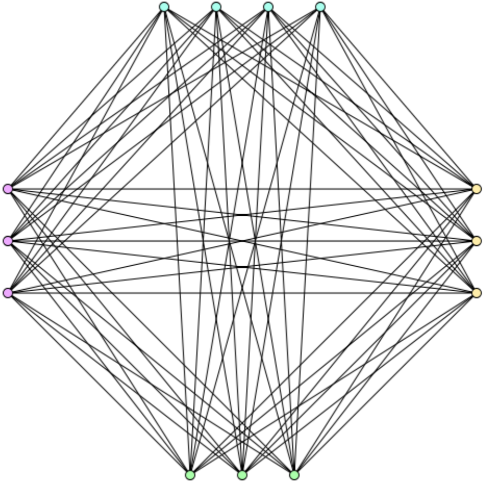


- $|V_i| = \left\lfloor \frac{n}{2} \right\rfloor$ or $\left\lceil \frac{n}{2} \right\rceil$
- triangle-free (K_3-free)
- maximum number of edges
- $t_2(n) = \left\lfloor \frac{n^2}{4} \right\rfloor$

Turán Graph $T_4(13)$



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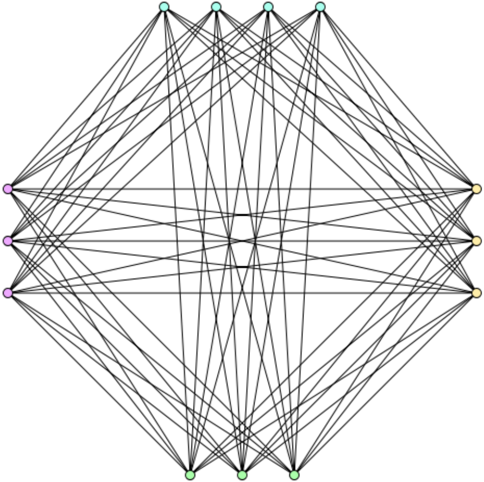

- 13 vertices, 4 clusters
- $|V_i| = 3$ or 4
- K_5-free
- maximum number of edges

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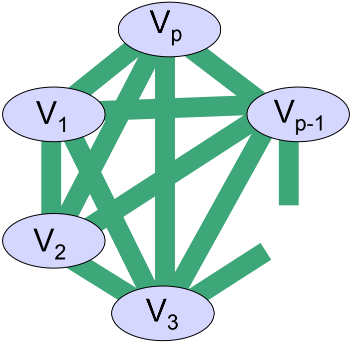

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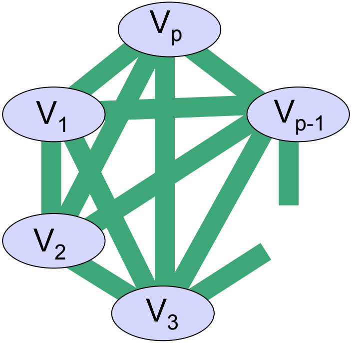

- n vertices, p clusters

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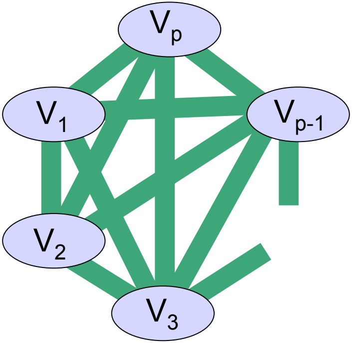

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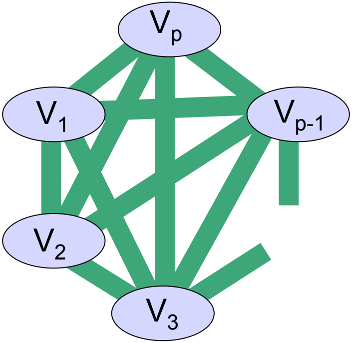

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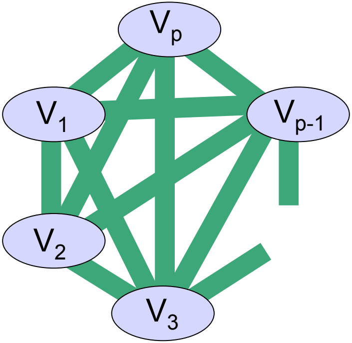

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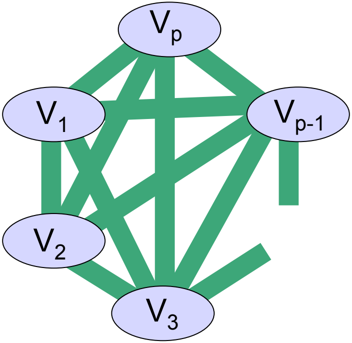

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- $|V_i| = \lfloor \frac{n}{p} \rfloor$ or $\lceil \frac{n}{p} \rceil$
- K_{p+1}-free
- maximum number of edges
- $t_p(n) = \left(1 - \frac{1}{p} + o(1)\right)\binom{n}{2}$

Some results about $\phi(n, H)$

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- Pikhurko-S. ['07]: $\phi(n, H) = t_{p-1}(n) + o(n^2)$, if $\chi(H) = p \geq 3$;
- Özkaya-Pearson ['12]: $\phi(n, H) = t_{p-1}(n)$, H edge-critical, $\chi(H) = p \geq 3$, n large.

Monochromatic H-Decompositions of Graphs

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$$\phi_k(n, H) = \max \{\phi_k(G, H) |\ v(G) = n\}$$

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First open instance is $H = K_r$

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- Ramsey Numbers are finite but notoriously difficult to calculate;
  - $R_2(3) = 6$, Greenwood and Gleason ['55];
  - $R_3(3) = 17$, Greenwood and Gleason ['55];
  - $R_2(4) = 18$, Greenwood and Gleason ['55];

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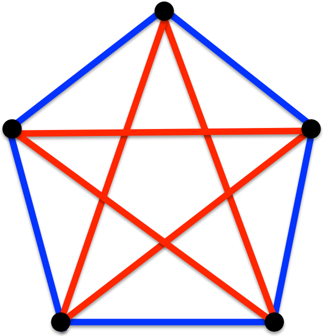

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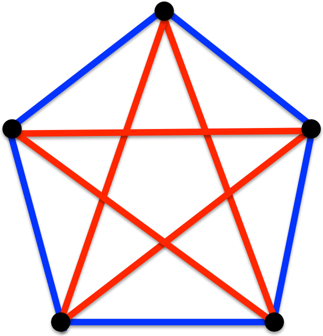

No monochromatic K_3

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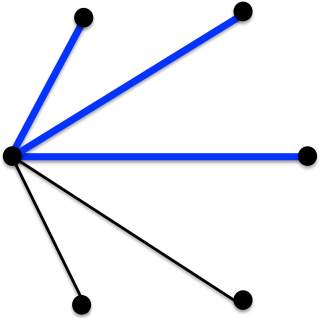

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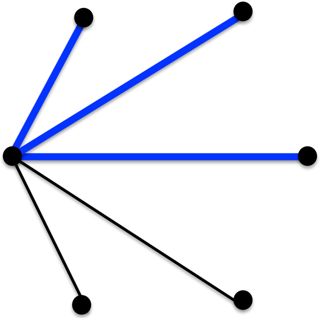

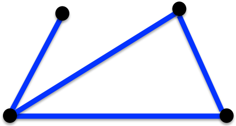

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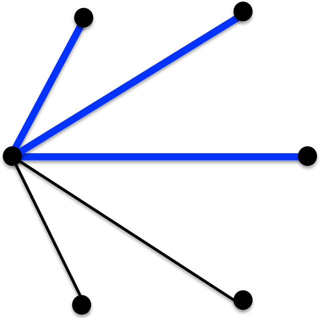

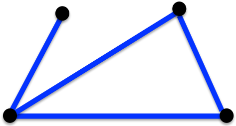

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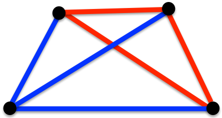

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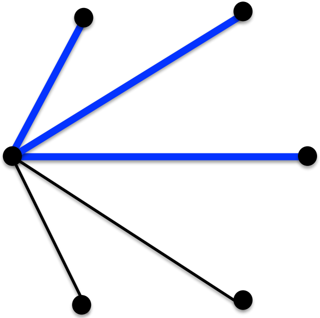

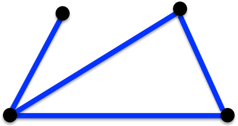

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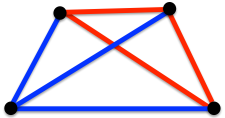

Red K_3

Thus $R_2(3) \leq 6$

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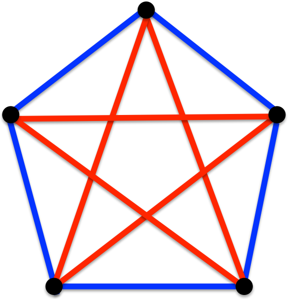

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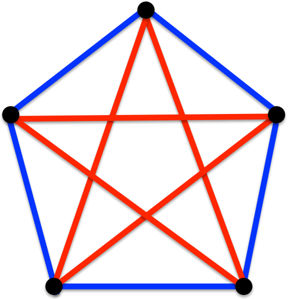

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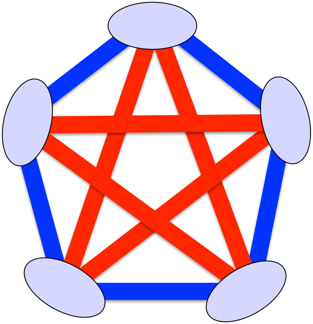

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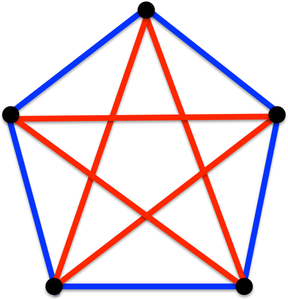

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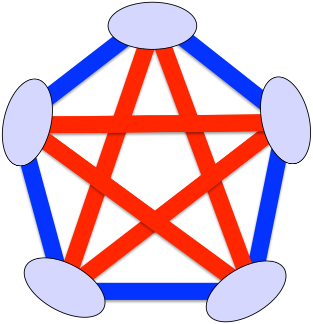

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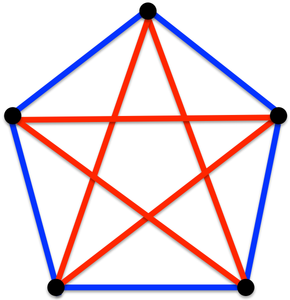

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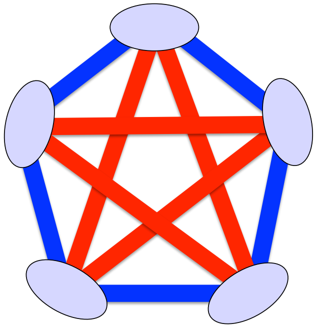

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$\phi_2(n, K_3) \geq t_5(n)$

Monochromatic K_3-Decompositions of Graphs

Theorem

Let $k \geq 2$, $R = R_k(3)$, then

$$\phi_k(n, K_3) = t_{R-1}(n) + o(n^2).$$

In particular,

$$\phi_2(n, K_3) = t_5(n) + o(n^2);$$

$$\phi_3(n, K_3) = t_{16}(n) + o(n^2).$$

Monochromatic K_r-Decompositions of Graphs

Theorem

Let $r \geq 4$, $k \geq 2$, $R = R_k(r)$ and n suficiently large, then

$$\phi_k(n, K_r) = t_{R-1}(n).$$

In particular, $\phi_2(n, K_4) = t_{17}(n)$.

Moreover, the only graph attaining $\phi_k(n, K_r)$ is the Turán graph $T_{R-1}(n)$.

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- $T_{R-1}(n)$ with this k-edge-coloring has no mono K_r;
- $\phi_k(n, K_r) \geq \phi_k(T_{R-1}(n), K_r) = t_{R-1}(n)$.

Proof of the upper bounds: Tools

- K_r-covering of G: set of edges whose removal results in a K_r-free graph;
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- K_r-packing of G: set of pairwise edge-disjoint K_r's;
- $\gamma_r(G)$ = maximum size of a K_r-packing of G.

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Theorem (Yuster '12)

$$\tau_3(G) \leq 2\gamma_3(G) + o(n^2);$$

$$\tau_r(G) \leq \left\lfloor \frac{r^2}{4} \right\rfloor \gamma_r(G) + o(n^2); \ \textit{for } r \geq 4 \ .$$

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  - $\ell > \frac{m}{2} \Rightarrow \phi_k(G, K_3) \leq \ell + e(G) - 3\ell \leq t_{R-1}(n) + \varepsilon n^2$ and we are done.

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- Therefore,

$$\begin{aligned} \sum_{i=1}^{k} \tau_3(G) &\leq \sum_{i=1}^{k} \left(2\gamma_3(G_i) + \frac{\varepsilon}{2k}n^2\right) \\ &\leq 2\ell + \frac{\varepsilon}{2}n^2 \\ &\leq m + \frac{\varepsilon}{2}n^2 \end{aligned}$$

Proof: $\phi_k(n, K_3) \leq t_{R-1}(n) + o(n^2)$

- By deleting at most $m + \frac{\varepsilon}{2}n^2$ edges we obtain G' without mono K_3

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- Suffices to show $\ell > \frac{m}{\binom{r}{2}-1}$

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- thus, $\ell > \frac{m}{\binom{r}{2}-1}$ and we are done

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- Therefore,

$$\begin{aligned} \sum_{i=1}^{k} \tau_r(G) &\leq \sum_{i=1}^{k} \left(\left\lfloor \frac{r^2}{4} \right\rfloor \gamma_r(G) + o(n^2) \right) \\ &\leq \left\lfloor \frac{r^2}{4} \right\rfloor \ell + o(n^2) \\ &\leq \frac{4}{5} m + o(n^2), \text{ since } r \geq 4. \end{aligned}$$

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Moreover, the only graph attaining the equality is the Turán graph $T_{R-1}(n)$.

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Proof: More technical, we need to know the structure of the Ramsey colorings

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ThankYou!