Matroids over rings

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4^{o} Dia de Combinatória, Aveiro

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Outline

This talk is on joint work with Luca Moci, `arXiv:1209.6571`.

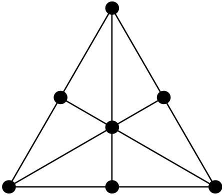
- Matroids
- Matroids over a ring
- Subtorus arrangements produce matroids over $\mathbb{Z}$
- Tropical geometry produces matroids over a valuation ring
- Duality and the Tutte invariant
- Speculations

Matroids

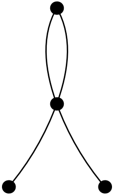
Matroids Whitney & others, ~'35 distil combinatorics from linear algebra.

An early perspective: axiomatize how (abstract) points can be contained in lines, planes, . . .

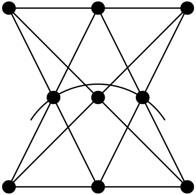
The pictures below are projective:



OK



Bad: $\{P, Q, R\}$ and $\{P, Q, S\}$ collinear $\Rightarrow$ all four collinear.



OK, despite Pappus! (nonrealizable)

Matroids

Matroids Whitney & others, ∼'35 distil combinatorics from linear algebra.

Many superficially unrelated definitions. (Birkhoff: "cryptomorphism".)

Definition

A matroid M on the finite ground set E assigns to each subset $A \subseteq E$ a rank $\mathrm{rk}(A) \in \mathbb{Z}_{\geq 0}$, such that: [. . .]

Main example: realizable matroids

Let $v_1, \ldots, v_n$ be vectors in a vector space V.

$$\mathrm{rk}(A) := \dim \mathrm{span}\{v_i : i \in A\}$$

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(0) $\mathrm{rk}(\emptyset) = 0$

(1) $\mathrm{rk}(A) \leq \mathrm{rk}(A \cup \{b\}) \leq \mathrm{rk}(A) + 1 \qquad \forall A \not\ni b$

(2) $\mathrm{rk}(A) + \mathrm{rk}(A \cup \{b, c\}) \leq \mathrm{rk}(A \cup \{b\}) + \mathrm{rk}(A \cup \{c\}) \qquad \forall A \not\ni b, c$

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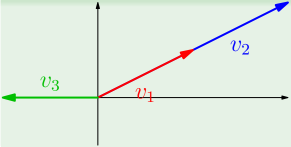
Example: a realizable matroid in full



A	$\emptyset$	1	2	12	3	13	23	123
$\mathrm{rk}(A)$	0	1	1	1	1	2	2	2

You might want to capture more than just the linear dependences:

Oriented matroids come from real vector configurations, and remember signs (e.g. in circuits). [Bland-las Vergnas]

Complex matroids come from complex configurations, and remember phases. [Anderson-Delucchi]

Valuated matroids come from configs over a field with valuation, and remember valuations. [Dress-Wenzel]

Arithmetic matroids come from configurations over $\mathbb{Z}$, and remember indices of sublattices. [D'Adderio-Moci]

Matroids over rings encompass these latter two.

(Compare matroids with coefficients [Dress].)

Enriched variants of matroids

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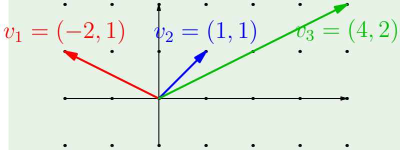
Let R be a commutative ring.

Let $v_1, \dots, v_n$ be a configuration of vectors in an R-module N.
We would like a system of axioms for the quotients $N/\langle v_i : i \in A\rangle$.

Realizable example



A	$\emptyset$	1	2	12
$M(A)$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/3$
A	3	13	23	123
$M(A)$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/8$	$\mathbb{Z}/2$	1

Matroids over rings: definition

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Main definition [F-Moci]

A matroid over R on the finite ground set E assigns to each subset $A \subseteq E$ a f.g. R module $M(A)$ up to $\cong$, such that

for all $A \subseteq E$ and $b, c \notin A$, there are elements

$$x = x(b,c), \quad y = y(b,c) \in M(A)$$

with

$$M(A) = M(A), \qquad M(A \cup \{b\}) \cong M(A)/\langle x\rangle,$$
$$M(A \cup \{c\}) \cong M(A)/\langle y\rangle, \qquad M(A \cup \{b,c\}) \cong M(A)/\langle x, y\rangle.$$

The maps between the modules $M(A)$ are not data!
This allows nonrealizability.

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(1) For all $A \not\ni b$, there is a surjection $M(A) \twoheadrightarrow M(A \cup \{b\})$ with cyclic kernel.

(2) For all $A \not\ni b, c$, there are four such maps forming a pushout

$$\begin{tikzcd} M(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M(A \cup \{b\}) \arrow[d] \\ M(A \cup \{c\}) \arrow[r] & M(A \cup \{b, c\}) \end{tikzcd}$$

(i.e. the square commutes and $\ker \searrow = \ker \downarrow + \ker \rightarrow$)

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This allows nonrealizability.

Matroids are matroids over fields

Theorem 1 (F-Moci)

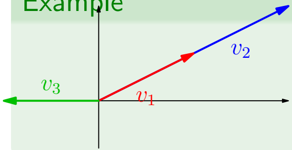
Matroids over a field $\mathbf{k}$ are equivalent to matroids*.

*if $M(E) = \emptyset$.

A f.g. $\mathbf{k}$-module is determined by its dimension $\in \mathbb{Z}$.

If $v_1, \ldots, v_n$ are vectors in $\mathbf{k}^r$,
the dimension of $\mathbf{k}^r/\langle v_i : i \in N\rangle$ is $r - \mathrm{rk}(A)$, the corank of A.

Example



A	$\emptyset$	1	2	12	3	13	23	123
$M(A)$	$\mathbb{R}^2$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	0	0	0

Note: The definition of matroids over $\mathbf{k}$ is blind to which field $\mathbf{k}$ is.
For realizability the choice matters.

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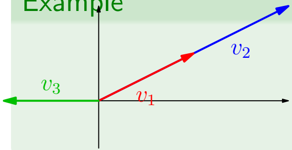

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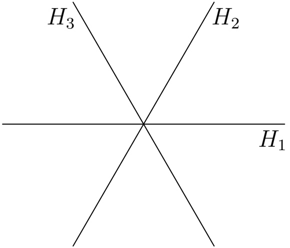
Application 1: hyperplane arrangement comb. & top.

Let $\mathcal{H} = \{H_1, \dots, H_n\}$ be hyperplanes in a vector space W, $\dim W = r$.



$\mathcal{H}$ has a matroid: $\mathrm{rk}(A) = \mathrm{codim} \bigcap_{i \in A} H_i$.

This is also the matroid of any dual vector configuration: $(v_i \in W^\vee)$ such that

$$H_i = \{x : \langle x, v_i \rangle = 0\}.$$

From the characteristic polynomial of $\mathcal{H}$, we get a lot of topology:

$$\chi_{\mathcal{H}}(q) = \sum_{A \subseteq E} (-1)^{|A|} q^{r - \mathrm{rk}(A)}$$

- $$\sum_k \dim H^k(W_{\mathbb{C}} \setminus \bigcup \mathcal{H}) q^k = (-q)^r \chi_{\mathcal{H}}(-1/q).$$
- $W_{\mathbb{R}} \setminus \bigcup \mathcal{H}$ has $(-1)^r \chi_{\mathcal{H}}(-1)$ components.

Etc.

Let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be hyperplanes in a vector space W, dim $W = r$.

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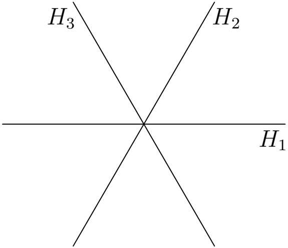

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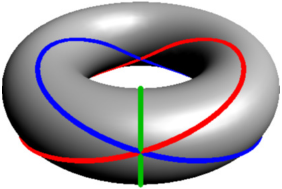
Subtorus arrangements

Now let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be codimension one tori in an r-dimensional torus T.
[De Concini-Procesi '10]



Subtori are dual to characters $u_i \in \mathrm{Char}(T)$:

$$H_i = \{x : u_i(x) = 1\}.$$

There is again a characteristic polynomial:

$$\chi_{\mathcal{H}}(q) = \sum_{A \subseteq E} (-1)^{|A|}\, m(A)\, q^{r - \mathrm{rk}(A)}.$$

Here

$$\begin{aligned} \mathrm{rk}(A) &= \ \mathrm{codim} \textstyle\bigcap_{i \in A} H_i = && \dim \mathrm{span}\{u_i : i \in A\} \\ m(A) &= \ \#\text{ components } \textstyle\bigcap_{i \in A} H_i = && [\mathbb{R}\{u_i\} \cap \mathrm{Char}(T) \ : \ \mathbb{Z}\{u_i\}] \end{aligned}$$

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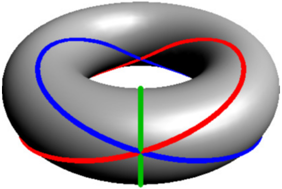

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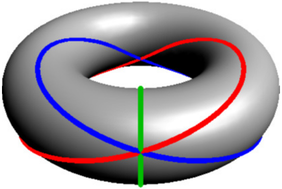

In terms of the characteristic polynomial

$$\chi_{\mathcal{H}}(q) = \sum_{A \subseteq E} (-1)^{|A|}\, m(A)\, q^{r - \mathrm{rk}(A)},$$

- The complex cohomology of a toric arrangement is given by

$$\sum_k \dim H^k(T \setminus \bigcup \mathcal{H}) q^k = (-q)^r \chi_{\mathcal{H}}(-(q+1)/q).$$

- $T \setminus \bigcup \mathcal{H}$ has $(-1)^r \chi_{\mathcal{H}}(0)$ components over the reals.

  Etc.

Arithmetic matroids

Definition ([Moci-D'Adderio])

An arithmetic matroid is a pair (M, m), where M is a matroid and $m : 2^E \to \mathbb{Z}_{>0}$ a *multiplicity* function, such that [complicated axioms]

We have a configuration $u_i \in \mathrm{Char}(T) \cong \mathbb{Z}^r$, and:

Theorem 2 (F-Moci)

Arithmetic matroids are matroids over $\mathbb{Z}$.

... almost. Arithmetic matroids forget the torsion structure:

$$\mathbb{Z}^r/\langle u_A \rangle = \mathbb{Z}^{r-d} \oplus F \qquad \Longrightarrow \qquad (M(A), m(A)) = (d, |F|)$$

where F is finite.

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Other appearances of matroids over $\mathbb{Z}$

In topology:

- Homology groups in quotients of spheres by finite groups [Hughes-Swartz].
- Maybe flows on simplicial complexes [Chmutov et al]?

This lies among many algebro-geometric applications of matroids:

moduli of hyp arrs [Hacking-Keel-Tevelev], compactifying fine Schubert cells [Lafforgue], classes of T-orbits on Grassmannians [F-Speyer], ...

Tropical geometry studies combinatorial "shadows" of algebraic varieties.

Two conics over $\mathbb{C}$ meet in four points [Bézout]

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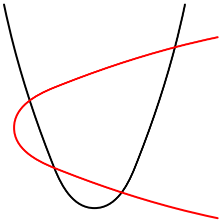

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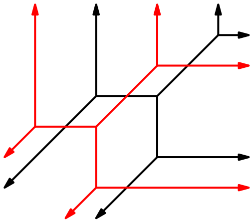

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Tropicalization

An algebraic variety $X \subseteq (\mathbf{k}^\times)^n$ has a tropicalization $\operatorname{Trop} X \subseteq \mathbb{R}^n$.

Easy case: If $(\mathbf{k}, \nu)$ has nontrivial valuation $\nu : \mathbf{k}^\times \to \mathbb{R}$, and $\mathbf{k} = \overline{\mathbf{k}}$, then $\operatorname{Trop} X = \overline{\nu(X)}$, coordinatewise.

A linear space $L \subseteq \mathbf{k}^n$ meets the torus in the hyperplane arrangement complement

$$L \cap (\mathbf{k}^\times)^n \subseteq (\mathbf{k}^\times)^n.$$

If ν is trivial, then $\operatorname{Trop} L$ is the fan whose cones are spanned by chains of *flats* of M.

Theorem (Speyer, '04)

There is a bijection

$$\{\textit{tropical linear spaces that are fans}\} \longleftrightarrow \{\textit{matroids}\}$$

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General tropical linear spaces

Proposition (Speyer, '04)

$$\{\textit{tropical linear spaces}\} \longleftrightarrow \left\{\begin{matrix}\textit{regular subdivisions} \\ \textit{of matroid polytopes}\end{matrix}\right\}$$

Definition; proposition (Dress-Wenzel, '91)

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The main axiom is a tropical Plücker relation for a Grassmannian: in

$$\{m(Abc) + m(Ade), m(Abd) + m(Ace), m(Acd) + m(Abe)\},$$

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Let (R, ν) be a valuation ring.

Theorem 3 (F-Moci)

A matroid over R gives a valuated matroid, i.e. a tropical linear space.

The values of m are the lengths over $R/\mathfrak{m}$ of the modules $M(A)$ with $|A| = d$.

But there's lots more data than that.
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$$\{\text{matroids over } R\} \xrightarrow{-\otimes_R S} \{\text{matroids over } S\}$$

Strategy

To understand matroids over a ring R (e.g. $\mathbb{Z}$):

1. What can their localizations be like? ($\Rightarrow$ valuation ring case)
2. When does a family of localizations come from a global matroid?

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Duality

Dual matroids arise from dual planar graphs, perpendicular subspaces, Gale dual vector configurations . . .

The dual M^* of a matroid M is given by

$$\operatorname{cork}_{M^*}(E \setminus A) = \operatorname{cork}_M(A) - r + |A|.$$

Let R be one of the following:

- a Prüfer domain, i.e. all localizations are 1-dim'l valuation rings (includes Dedekind domains);
- a local Noetherian ring.

Theorem (F-Moci)

Matroids over R have well-defined duals.

The construction is by dualizing a resolution of $\ker(M(\emptyset) \to M(A))$.

In the Dedekind case, $\quad M^*(E \setminus A) \cong \operatorname{Ext}^1(M(A), R)$
up to projective modules of rank difference $-r + |A|$.

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The construction is by dualizing a resolution of $\ker(M(\emptyset) \to M(A))$.

In the Dedekind case, $M^*(E \setminus A) \cong \mathrm{Ext}^1(M(A), R)$ up to projective modules of rank difference $-r + |A|$.

Duality

Dual matroids arise from dual planar graphs, perpendicular subspaces, Gale dual vector configurations . . .

The dual M^* of a matroid M is given by

$$\mathrm{cork}_{M^*}(E \setminus A) = \mathrm{cork}_M(A) - r + |A|.$$

Let R be one of the following:

- a Prüfer domain, i.e. all localizations are 1-dim'l valuation rings (includes Dedekind domains);
- a local Noetherian ring.

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The Tutte polynomial (after [Brylawski])

The deletion $M \setminus i$ is the restriction of M to sets $A \not\ni i$, and
the contraction M/i is the restriction of M to sets $A \ni i$.

Define the Tutte-Grothendieck ring to be the free group on $\{T_M : M \text{ a matroid}\}$ modulo relations

$$T_M = T_{M\setminus i} + T_{M/i},$$

and product $T_M T_{M'} = T_{M \oplus M'}$.

T_M is the Tutte polynomial of M, with many important evaluations (e.g. characteristic poly, chromatic poly).

Theorem (Crapo, Brylawski)

The Tutte-Grothendieck ring is $\mathbb{Z}[x-1, y-1]$, with

$$T_M = \sum_{A \subseteq E} (x-1)^{\operatorname{corank}_M(A)} (y-1)^{\operatorname{corank}_{M^*}(E \setminus A)}$$

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The Tutte-Grothendieck ring is $\mathbb{Z}[x-1, y-1]$, *with*

$$T_M = \sum_{A\subseteq E} (x-1)^{\operatorname{corank}_M(A)} (y-1)^{\operatorname{corank}_{M^*}(E\setminus A)}$$

The Tutte polynomial for matroids over R

Let R be a Dedekind domain.

Let $\mathbb{Z}[R\text{-Mod}]$ be the monoid ring of fin. gen. R-modules up to $\cong$ under direct sum. $\quad u^N u^{N'} = u^{N \oplus N'}$

Theorem (F-Moci)

The Tutte-Grothendieck ring of matroids over R injects into $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$, with

$$\textit{class of } M = \sum_{A \subseteq E} X^{M(A)}\, Y^{M^*(E \setminus A)}$$

It's a proper injection since $M(A)$ and $M^*(E \setminus A)$ have the same torsion part.

Some specializations:

- The characteristic polynomial of a subtorus arrangement
- The Tutte quasipolynomial of [Brändén-Moci]

Future work

- Other axiom systems: polytopes, bases, circuits, ...?
- Are duals always well-defined?
- Which rings have good characterizations of realizability?
- What's the extra data over a DVR?
  (maybe: convex hulls in buildings [Joswig-Sturmfels-Yu])
- Implications for algebraic matroids?

Thank you!

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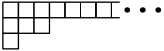
Thank you!

A bit more on DVRs

There is a bijection between
finitely generated modules over a DVR
& partitions allowing infinite parts.

Example

$N_\lambda = R \oplus R/\mathfrak{m}^3 \oplus R/\mathfrak{m}$

$\lambda =$ 

Theorem (Hall, . . .)

The number of exact sequences

$$0 \longrightarrow N_\lambda \longrightarrow N_\nu \longrightarrow N_\mu \longrightarrow 0$$

up to $\cong$ *of sequences is the LR coeff* $c^\nu_{\lambda\mu}$ *(or its infinite-rows analog).*

So, quotients by one element give the Pieri rule.

Lemma, en route to Theorem 3

M is a 1-element matroid over R $\Longleftrightarrow$
$M(\emptyset)$ has at most one box more in each column than $M(1)$.