Laplacian and signless Laplacian spectra of graphs having vertex subsets with common neighborhood properties

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- $A(G)$ is the adjacency matrix of the graph G.
- $\mathrm{D}(G)$ is the $n \times n$ diagonal matrix of the vertex degrees of G.

- $L(G) = \mathrm{D}(G) - A(G)$ and $Q(G) = \mathrm{D}(G) + A(G)$ are the Laplacian and signless Laplacian matrix of G, respectively.

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- As all these matrices are real and symmetric their eigenvalues are real (nonnegative in case of Laplacian and signless Laplacian matrices).
- The second smallest eigenvalue of $L(G)$ is called the algebraic connectivity of G.

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- the *degree* of a cluster is the cardinality of the shared set of neighbors (i.e. the common degree of each vertex in the cluster).
- an l-cluster is a cluster of degree l.

Motivation

I. Faria, in [3], introduced the following result about Laplacian and signless Laplacian eigenvalues of graphs with leaves.

Theorem [3]

Let p and q be the number of leaves of G and the number of neighbors associated to these leaves, respectively. Then 1 is a Laplacian (signless Laplacian) eigenvalue of G with multiplicity at least $p - q \geq 0$.

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- $V(G^k) = V(G)$ and $E(G^k) = E(G) \cup E(G_k)$.

An example

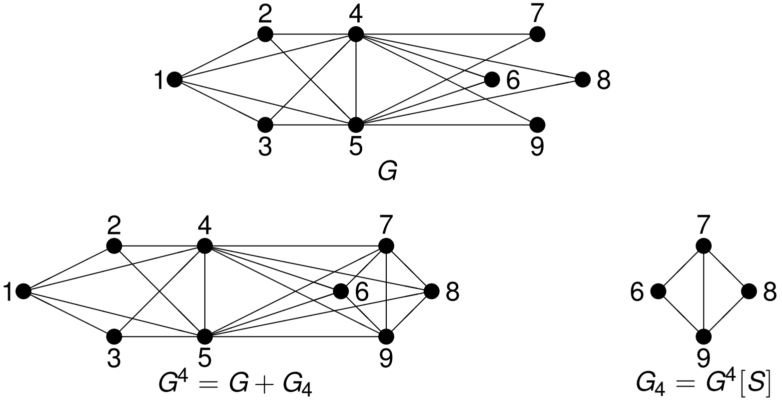

Figure: A graph G with a 2-cluster of order 4, $S = \{6, 7, 8, 9\}$ and the graphs $G^4 = G + G_4$, such that $G_4 = G^4[S]$.

Main Results

- For a graph G of order n, with k pairwise co-neighbors the number of shared neighbors is a Laplacian and a signless Laplacian eigenvalue of G with multiplicity at least $k-1$.

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- For a graph G of order n, with k pairwise co-neighbors the number of shared neighbors is a Laplacian and a signless Laplacian eigenvalue of G with multiplicity at least $k - 1$.
- determination of Laplacian and signless Laplacian eigenvalues of G^k (for which the induced subgraph G_k must be p-regular in the signless Laplacian case).

- assuming that S is an l-cluster of order k, and

$$\beta \neq 0$$

($\beta \neq 2p$) is a Laplacian (signless Laplacian) eigenvalue of G_k, it is deduced that $l + \beta$ is a Laplacian (signless Laplacian) eigenvalue of G^k.

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- Furthermore, in the Laplacian spectrum case, we may conclude that at least

$$n - k + 1$$

 Laplacian eigenvalues of G are also eigenvalues of G^k.

Theorem [1]

Let G be a graph with an l-cluster S of order k. Then l is a Laplacian and a signless Laplacian eigenvalue of G with multiplicity at least $k-1$.

An Application: $K_{r,s}$

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Let $K_{r,s}$ a complete bipartite graph. It follows that each color class of vertices is a vertex subset of pairwise co-neighbors. Therefore, despite the Laplacian spectrum of $K_{r,s}$ be well known, we may conclude that

- r is a Laplacian eigenvalue with multiplicity at least $s-1$
- s is a Laplacian eigenvalue with multiplicity at least $r-1$.

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then the unknown eigenvalue is $r+s$. Thus

$$\sigma_L(K_{r,s}) = \sigma_Q(K_{r,s}) = \{0, r^{[s-1]}, s^{[r-1]}, r+s\}.$$

Laplacian Eigenvalues of G^k

We present the theorem that shows how the Laplacian eigenvalues of G^k are modified in function of G_k.

Theorem [1]

Let G be a graph with an l-cluster S of order k. Assume that G_k is a connected graph such that $V(G_k) = S$, $G^k = G + G_k$ and

$$\Lambda = \{l + \beta : \beta \in \sigma_L(G_k) \setminus \{0\}\}$$

is a multiset. Then $\sigma_L(G^k)$ overlaps $\sigma_L(G)$ in $n - k + 1$ places and the elements of Λ are the remaining eigenvalues in $\sigma_L(G^k)$.

Some Consequences

Taking into account that a graph G is called Laplacian integral if its Laplacian eigenvalues are all integers, it is immediate to conclude:

Corollary [1]

Let G be a graph with a cluster S of order k, and G_k a connected graph such that $V(G_k) = S$. If G and G_k are Laplacian integral graphs, then $G^k = G + G_k$ is also Laplacian integral.

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of some families of graphs.

Definition
The Laplacian spread of a graph is the difference between the largest Laplacian eigenvalue and the algebraic connectivity.

Let $G = K_{r,s}$ on $r + s$ vertices and a connected graph G_r, such that

$$V(G_r) = S,$$

(an s-cluster of order r in G), then we may conclude the following result.

Theorem [1]

If $r \leq s$, $G = K_{r,s}$ and G_r is a connected graph defined on the vertex subset of r pairwise co-neighbors of G, then the graphs G and $G^r = G + G_r$ have

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If $r \leq s$, $G = K_{r,s}$ and G_r is a connected graph defined on the vertex subset of r pairwise co-neighbors of G, then the graphs G and $G^r = G + G_r$ have

- the same largest Laplacian eigenvalue $r + s$,
- the same algebraic connectivity r,
- and the same Laplacian spread s.

Signless Laplacian Eigenvalues of G^k

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- A good survey on this topic was published in [5].
- Herein, we extend this concept also to signless Laplacian matrices.

Given a graph G, an eigenvalue $\lambda \in \sigma_Q(G)$ is *non-main* if the corresponding eigenspace is orthogonal to the all one vector.

Theorem [1]

Let G be a graph with an l-cluster $S = \{v_i\}_{i=1}^{k}$ of order k. If G_k is a graph such that $V(G_k) = S$ and $G^k = G + G_k$, then $\sigma_Q(G^k)$ includes the multiset

$$\{l + \beta : \beta \in \sigma_Q\left(G_k\right) \text{ and it is non-main}\}.$$

Furthermore, any main eigenvalue γ of $Q(G_k)$, with multiplicity $m > 1$, produces an eigenvalue $l + \gamma$ of $Q(G^k)$, with multiplicity at least $m - 1$.

As immediate consequence of previous theorem, we have the following corollary.

Corollary [1]

If G_k is a p-regular graph defined on an l-cluster of order k of a graph G, and $G^k = G + G_k$, then $\sigma_Q(G^k)$ includes the multiset

$$\{l + \beta : \beta \in \sigma_Q(G_k) \setminus \{2p\}\}.$$

Note that if G_k is p-regular $2p$ is an eigenvalue of $Q(G_k)$ with the all one vector as a corresponding eigenvector. Therefore the only main eigenvalue is $2p$.

Teorema [1]

Consider the complete bipartite graph $G = K_{r,s}$, with vertex set $\{v_i\}_{i=1}^{r+s}$ and G_r a $p-$regular graph defined on the s-cluster of order r, $\{v_1, \ldots, v_r\}$ of G. If $G^r = G + G_r$, then

$$\{r^{[s-1]}\} \subset \sigma_Q(G) \cap \sigma_Q(G^r)$$

and the remainder signless Laplacian eigenvalues of G^r are the elements of the multiset

$$\{s + \gamma : \gamma \in \sigma_Q(G_r) \setminus \{2p\}\} \cup \left\{ \frac{r + s + 2p \pm \sqrt{(r + s + 2p)^2 - 8pr}}{2} \right\}.$$

Conclusions and Open Problems

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- Study cases for which the "overlapping" of the spectra can hold.
- In the signless laplacian case consider an arbitrary graph G_k instead of a p-regular graph.

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