Counting Spectral Radii of Matrices with Positive Entries

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$$|S + T| \geq \min\{p, |S| + |T| - 1\}.$$

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For $S + T \neq \mathbb{Z}_p$, one finds w for which this dim is at least $|S| + |T| - 1$.

Erdős-Heilbronn

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Let $\emptyset \neq A \subseteq \mathbb{Z}_p$, for a prime p. Let $2\hat{}A := \{a + b : a, b \in A, a \neq b\}$.

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$$|2\hat{}A| \geq \min\{p, 2|A|-3\}.$$

Proved in 1994 by Dias da Silva and Hamidoune: if $A \subseteq F$,

$$|n\hat{}A| \geq \min\{p(F), n|A|-n^2+1\}$$

where $p(F) = p$ if p is the characteristic of F, or $p(F) = \infty$ if the characteristic is zero.
Also used techniques of multilinear algebra.

Extensions: Noga Alon, M. Nathanson and I. Ruzsa in 1996, and others.

Erdős-Szemerédi

Erdős-Szemerédi Conjecture

No $A \subseteq \mathbb{N}$ will yield values of $|A + A|$ and $|A \cdot A|$ that are both "small". Meaning: for $A \subset \mathbb{N}$ a finite set, we have that for all $\delta < 1$,

$$\max\{|A + A|, |A \cdot A|\} \gg |A|^{1+\delta}.$$

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Best δ so far: Solymosi, 2009, $\delta = 1/3$.

$|A + A|$ is smallest for arithmetic progressions, $|A.A|$ is smallest for geometric progressions.

Spectral radii

Spectral radius of a matrix M:

$$\rho(M) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } M\}.$$

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Let $M \in (\mathbb{R}_0^+)^{n\times n}$, $M \neq 0$.

By the Perron-Frobenius theorem, the spectral radius of M is a positive eigenvalue of M.

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For $A \subseteq \mathbb{R}_0^+$, finite, define

$$\Omega_n(A) = \{\rho(M) \ : \ M \in A^{n\times n}\},$$

We are now interested in lower bounds for $|\Omega_n(A)|$.

Relation to previous problems

Proposition

We have $nA \subseteq \Omega_n(A)$ and $|A^{\times n}| \leq |\Omega_n(A \cup \{0\})|$.

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$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \end{bmatrix} \qquad \begin{bmatrix} 0 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-1} \\ a_n & 0 & \cdots & 0 \end{bmatrix}$$

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$$\rho = a_1 + \ldots + a_n \qquad \rho = \sqrt[n]{a_1 \ldots a_n}$$

Therefore

$$|\Omega_n(A \cup \{0\})| \geq \max\{|nA|, |A^{\times n}|\}.$$

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Therefore

$$|\Omega_n(A \cup \{0\})| \geq \max\{|nA|, |A^{\times n}|\}.$$

Proposition ($n = 2$)

$$|\Omega_2(A)| \geq \max\{|A + A|, |A \cdot A|\}.$$

Lower bounds with set restrictions

For $A \subseteq \mathbb{N}$, finite, define A' as the set obtained dividing all elements by the g.c.d.

$$|A| = |A'| \quad \text{and} \quad |\Omega_n(A)| = |\Omega_n(A')|.$$

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Let $\mathbb{P}$ be the set of prime numbers, and $p \in \mathbb{P}$. Set

$$\pi_A = \max\left\{ \frac{|A' \cap p * \mathbb{N}|}{|A'|} \ : p \in \mathbb{P},\ (A' \cap p * \mathbb{N}) \setminus p^2 * \mathbb{N} \neq \emptyset \right\},$$

or 0 if there are no such primes.

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Example 1: $A = \{3, 4, 5, 8, 9, 12\} = A'$, $\pi_A = 1/2$ attained with $p = 3$ but not $p = 2$.

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Example 1: $A = \{3, 4, 5, 8, 9, 12\} = A'$, $\pi_A = 1/2$ attained with $p = 3$ but not $p = 2$.

Example 2: $A = \{4, 9, 36\} = A'$, $\pi_A = 0$.

Lower bounds with set restrictions

Theorem

For any finite set $A \subseteq \mathbb{N}$ with $\pi_A > 0$,

$$|\Omega_n(A \cup \{0\})| \geq \pi_A |A|^{n-1} \geq |A|^{n-2}.$$

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If A is a geometric progression with prime ratio p, then

$$A' = \{a, pa, p^2 a, \ldots, p^k a\}$$

where $p \not| a$. So $\pi_A = (|A| - 1)/|A|$.

In this case we get $|\Omega_n(A \cup \{0\})| \geq |A - 1||A|^{n-2}$.

Matrices with positive real entries

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Take $A = \{a_1 < \ldots < a_k\}$ and the sequence of sums:

$$\begin{array}{lllll} a_1 + a_1, & a_1 + a_2, & a_1 + a_3, & \ldots & a_1 + a_k, \\ & a_2 + a_k, & a_3 + a_k, & \ldots & a_k + a_k. \end{array}$$

Each entry goes through $k - 1$ increments, so the length is $2(k-1) + 1$.

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With positive matrices, the spectral radius increases if one entry increases.

The same argument gives us the lower bound

$$|\Omega_n(A)| \geq n^2(|A| - 1) + 1.$$

Using matrix structure

Proposition (B. Schwartz, 1964)

Let $M = [m_{ij}] \in (\mathbb{R}^+)^{n \times n}$ and let $v = (v_1, \ldots, v_n)$ be its Perron vector. Suppose that for some indices r, s, t, with $s < t$,

$$(m_{rs} - m_{rt})(v_s - v_t) < 0.$$

Then the matrix obtained from M exchanging entries m_{rs} and m_{rt} has a greater spectral radius than M.

Our technique

Create a sequence of matrices, with entries in A, with strictly increasing spectral radius, using

- increments in entries and
- exchanges in row elements.

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In the matrices we will get, we have that:

to larger row sums correspond larger Perron vector entries.

However, this is not true in general.

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \rho = 3.00 \quad v = (1.0, 1.0, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \rho = 3.30 \quad v = (1.3, 1.0, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \rho = 3.41 \quad v = (1.4, 1.0, 1.0)$$

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Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad \rho = 3.65 \quad v = (1.4, 1.3, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \rho = 3.73 \quad v = (1.4, 1.4, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \quad \rho = 4.00 \quad v = (1.0, 1.0, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \quad \rho = 4.23 \quad v = (1.2, 1.0, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \quad \rho = 4.41 \quad v = (1.4, 1.0, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \quad \rho = 4.60 \quad v = (1.4, 1.2, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \rho = 4.73 \quad v = (1.4, 1.4, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad \rho = 4.79 \quad v = (1.3, 1.3, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix} \quad \rho = 5.00 \quad v = (1.0, 1.0, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix} \quad \rho = 5.19 \quad v = (1.2, 1.0, 1.0)$$

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A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix} \quad \rho = 5.51 \quad v = (1.3, 1.2, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix} \quad \rho = 5.56 \quad v = (1.3, 1.3, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \quad \rho = 5.62 \quad v = (1.3, 1.4, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \quad \rho = 5.76 \quad v = (1.2, 1.2, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \rho = 6.00 \quad v = (1.0, 1.0, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \rho = 6.16 \quad v = (1.2, 1.0, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \rho = 6.24 \quad v = (1.2, 1.0, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \rho = 6.54 \quad v = (1.2, 1.2, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 1 & 3 \\ 3 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} \quad \rho = 6.63 \quad v = (1.2, 1.2, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 1 & 3 \\ 3 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \rho = 6.70 \quad v = (1.2, 1.2, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 1 & 3 \\ 3 & 3 & 1 \\ 2 & 3 & 1 \end{bmatrix} \quad \rho = 6.73 \quad v = (1.1, 1.2, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix} \quad \rho = 7.00 \quad v = (1.0, 1.0, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 2 & 3 & 3 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix} \quad \rho = 7.16 \quad v = (1.2, 1.0, 1.0)$$

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$$\begin{bmatrix} 3 & 2 & 3 \\ 2 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix} \quad \rho = 7.61 \quad v = (1.2, 1.2, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 3 & 3 \end{bmatrix} \quad \rho = 7.65 \quad v = (1.1, 1.2, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 2 & 3 \\ 3 & 3 & 2 \\ 3 & 3 & 1 \end{bmatrix} \quad \rho = 7.73 \quad v = (1.1, 1.1, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 2 & 3 & 3 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix} \quad \rho = 8.00 \quad v = (1.0, 1.0, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 3 & 3 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix} \quad \rho = 8.27 \quad v = (1.1, 1.0, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix} \quad \rho = 8.65 \quad v = (1.1, 1.1, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 2 \end{bmatrix} \quad \rho = 8.69 \quad v = (1.1, 1.1, 1.0)$$

A small example

Let $A = \{1, 2, 3\}$, $n = 3$

$$\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \quad \rho = 9.00 \quad v = (1.0, 1.0, 1.0)$$

An improved lower bound

Theorem

Keeping the previous notation, we have the following lower bounds for $|\Omega_n(A)|$ if $n, |A| > 1$ and $\max\{n, |A|\} > 2$.

If $|A| < n$:

$$n\frac{(|A|-1)|A|(2|A|-1)}{3} + \\ +n(|A|-1)(|A|-2)(n-|A|) + n|A|(n-|A|+1) - n + 1.$$

If $|A| \geq n$:

$$n\frac{(n-1)n(2n-1)}{3} + n^3(|A|-n) + n^2 - n + 1.$$

The formulas coincide if $|A| = n$.

Final remarks

It should be true that $|\Omega_n(A)| \leq |\Omega_{n+1}(A)|$.

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Can we find $v_1, v_2 \in A^{n\times 1}$ and $a \in A$ such that if $\rho(M) < \rho(M')$, then

$$\rho\left(\begin{bmatrix} M & v_1 \\ v_2^T & a \end{bmatrix}\right) < \rho\left(\begin{bmatrix} M' & v_1 \\ v_2^T & a \end{bmatrix}\right)?$$

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It should be true that $|\Omega_n(A)| \leq |\Omega_{n+1}(A)|$.
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$$\rho\left(\begin{bmatrix} M & v_1 \\ v_2^T & a \end{bmatrix}\right) < \rho\left(\begin{bmatrix} M' & v_1 \\ v_2^T & a \end{bmatrix}\right)?$$

Not true in general:

$$\rho\left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\right) = 2.414 < \rho\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right) = 2.618,$$

$$\rho\left(\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix}\right) = 4.791 > \rho\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix}\right) = 4.732.$$

Some conjectures

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If A is a geometric progression with $|A| = k$, then, in most cases

$$|\Omega_2(A)| = \frac{1}{2}(2k^3 - k^2 + k),$$

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Conjecture

Let $B \subseteq \mathbb{R}^+$, with $|B| = k - 1$. Then $|\Omega_2(B \cup \{0\}|)| \leq r_k$.