Torsion pairs in negative Calabi-Yau triangulated categories

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April 2015

Motivation

- **k** is an algebraically closed field
- T is a **k**-linear triangulated category
- $m \in \mathbb{Z}$

Definition

T is *m-Calabi-Yau* (m-CY) if there is a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathsf{T}}(s, t) \simeq D\,\mathrm{Hom}_{\mathsf{T}}(t, \Sigma^m s),$$

for every $s, t \in \mathsf{T}$.

Notion important in:

- Theoretical physics,
- Algebraic and sympletic geometry,
- Representation theory: cluster-tilting theory.

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Aim: To understand the structure of negative CY triangulated categories.

One can consider:

- generators: tilting/silting objects, cluster-tilting objects, simple-minded collections, etc.
- particular subcategories: **torsion pairs**, t-structures, co-t-structures.

In tilting theory: torsion pairs give ways of comparing different module categories.

In cluster-tilting theory: torsion pairs are a generalisation of cluster-tilting objects.

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Torsion pairs

- Every subcategory is full and closed under summands.

Definition

A pair of subcategories (X, Y) is a *torsion pair* if:

1. $\mathrm{Hom}_{\mathsf{T}}(\mathsf{X}, \mathsf{Y}) = 0$, and
2. $\forall t \in \mathsf{T}, \exists x \to t \to y \to \Sigma x$ with $x \in \mathsf{X}, y \in \mathsf{Y}$.

Proposition (Iyama-Yoshino)

If (X, Y) *is a torsion pair then*

$$\mathsf{Y} = \mathsf{X}^{\perp} := \{t \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(\mathsf{X}, t) = 0\}.$$

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The categories

Setup

- Q: Dynkin quiver of type A_n
- kQ: the path algebra of Q
- $D^b(kQ)$: bounded derived category of the module category over kQ
- Σ: the shift functor
- τ: the Auslander-Reiten (AR) translate
- $m \in \mathbb{Z} \setminus \{0, 1\}$

We will consider the orbit categories:

$$B_m(kQ) := D^b(kQ)/\tau\Sigma^m$$

- $B_m(kQ)$ is a $(1-m)$-CY triangulated category.

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The *Auslander–Reiten quiver* of an algebra A has as vertices the indecomposable A-modules and as arrows the irreducible maps.

Example

$Q : \bullet \longrightarrow \bullet \longrightarrow \bullet$

AR-quiver of $\mathrm{mod}(\mathbf{k}Q)$:

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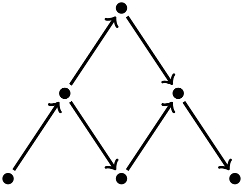
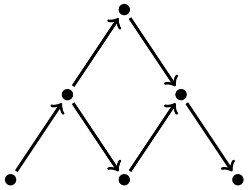


AR-quiver of the derived category

$D^b(\mathbf{k}A_3)$:



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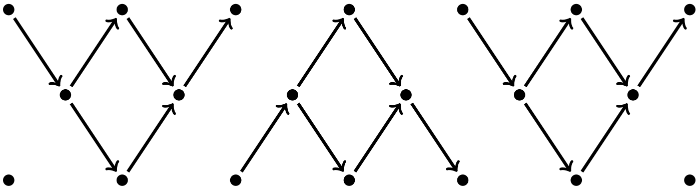

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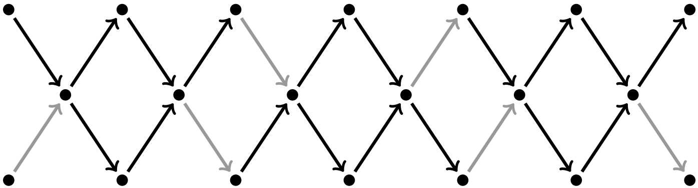
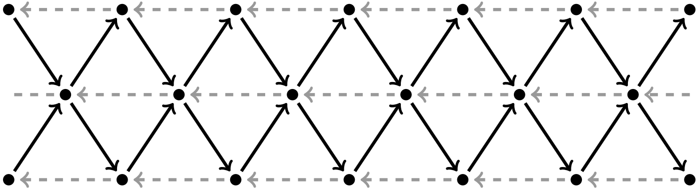


AR-quiver of the derived category

$D^b(\mathbf{k}A_3)$ - the AR-translate τ:



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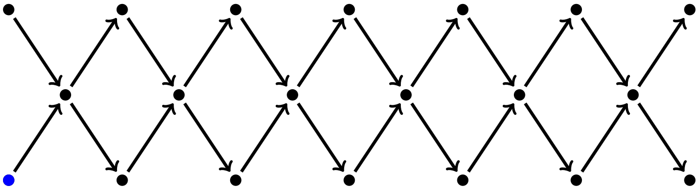

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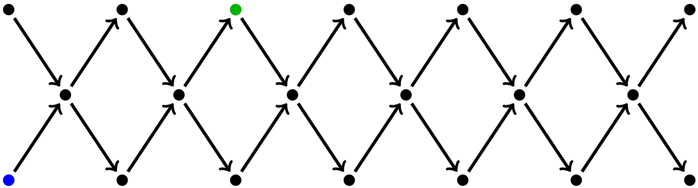

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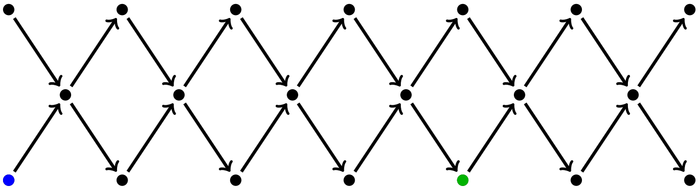

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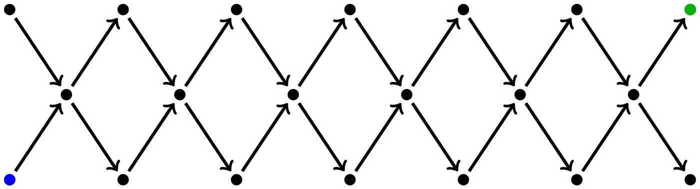

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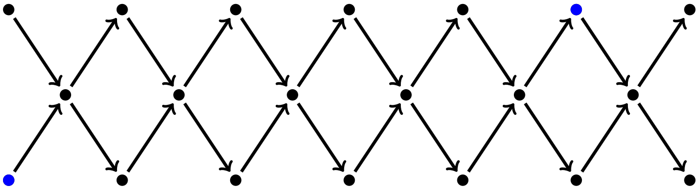

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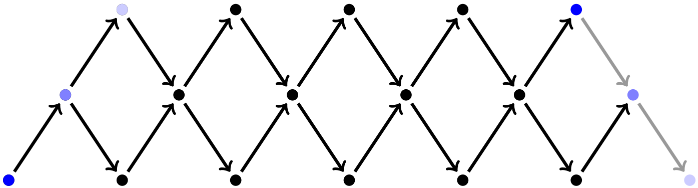

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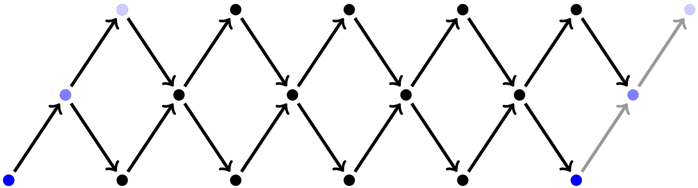

Combinatorial model of $B_m(\mathbf{k}A_n)$

P polygon with $N = \begin{cases} (n+1)|m| - 2 & \text{if } m \geqslant 2 \\ (n+1)|m| + 2 & \text{if } m \leqslant -1 \end{cases}$ vertices.

Definition

Let $i, j (i < j)$ be two vertices of P. The pair $\{i, j\}$ is a *m-diagonal* if $\exists k \in \{1, \ldots, n\}$ such that:

$$1 - km = \begin{cases} i - j & \text{if } m \geqslant 2 \\ j - i & \text{if } m \leqslant -1. \end{cases}$$

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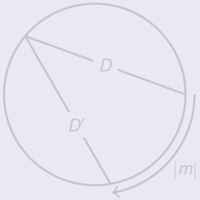
Let $\Gamma(m,n) = (\Gamma_0, \Gamma_1, \tau)$ be the stable translation quiver defined by:

- $\Gamma_0 =$ the set of m-diagonals of P.
- $\Gamma_1 : D \to D'$ if:



- $\tau(\{i,j\}) = \{i - |m|, j - |m|\}$.

Proposition (Caldero-Chapoton-Schiffler, Baur-Marsh, CS)

The AR-quiver of $B_m(A_n)$ and $\Gamma(m,n)$ are isomorphic as stable translation quivers.

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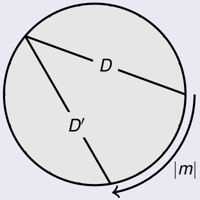

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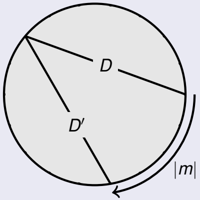

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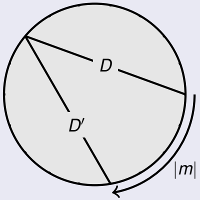

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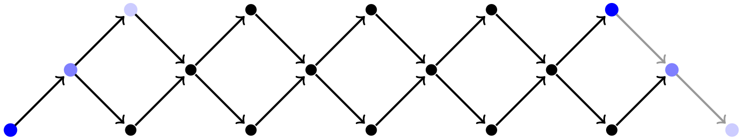
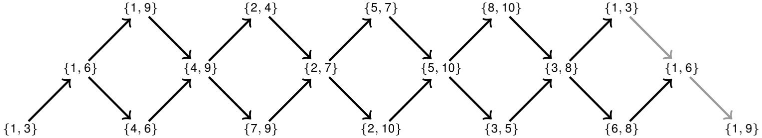
Example in negative CY case

$\mathsf{B}_3(\mathbf{k}A_3)$:



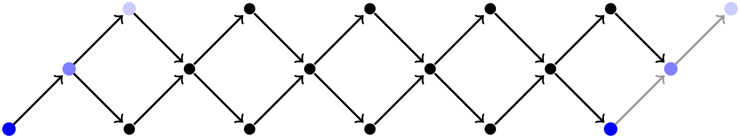
Example in negative CY case

$\Gamma(3, 3)$:



Example in positive CY case

$\mathsf{B}_{-2}(\mathbf{k}A_3)$:



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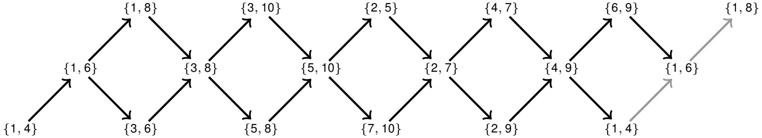
$\Gamma(-2,3)$:



Ptolemy arcs

Definition

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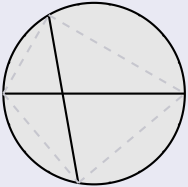

- Ptolemy arcs of class II:

Ptolemy arcs

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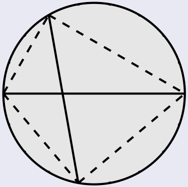

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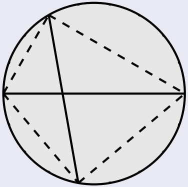

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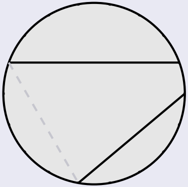

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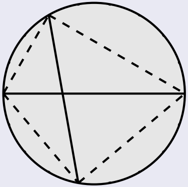

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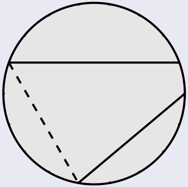

Classification of torsion pairs in $\mathsf{B}_m(A_n)$

Theorem (CS-Pauksztello, Holm-Jørgensen-Rubey for $m = -1$)

- X *subcategory of* $\mathsf{B}_m(A_n)$
- $\mathscr{X}$ *corresponding set of m-diagonals*

Then $(\mathsf{X}, \mathsf{X}^{\perp})$ *is a torsion pair if and only if:*

	$m \leqslant -1$	$m \geqslant 2$
$\mathscr{X}$ *closed under* ***admissible*** *Ptolemy arcs of class*	*I*	*I and II*

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- $\mathscr{X}$ *corresponding set of m-diagonals*

Then $(\mathrm{X}, \mathrm{X}^{\perp})$ *is a torsion pair if and only if:*

	$m \leqslant -1$	$m \geqslant 2$
$\mathscr{X}$ *closed under* ***admissible*** *Ptolemy arcs of class*	*I*	*I and II*

Classification of torsion pairs in $\mathsf{B}_m(A_n)$

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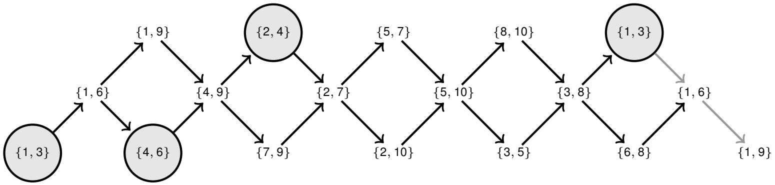
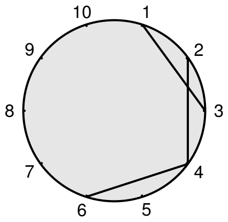
Example in $B_3(\mathbf{k}A_3)$





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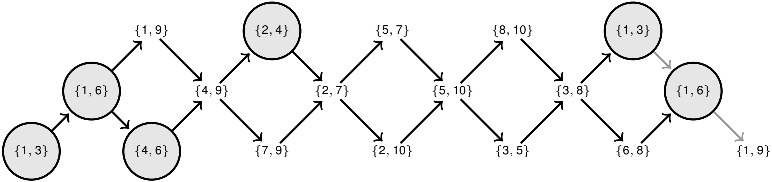

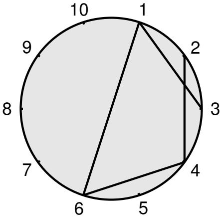

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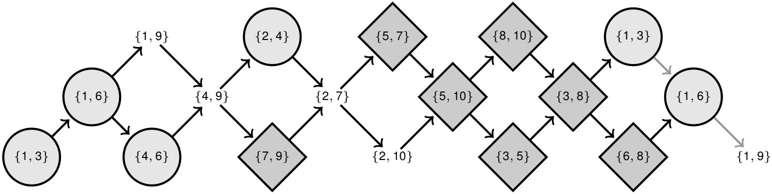

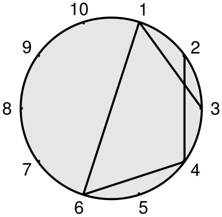

Thank you!