Regular and chiral hypertopes

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The genesis of the theory of hypertopes

Idea: Extend the theory of polytopes to more general objects by dropping linear condition.

Regular polytopes	$\equiv$	Thin residually connected regular geometries with linear diagram

A "hypertope" is a generalization of a "polytope" as a "hypermap" is a generalization of a "map".

- Definition of a hypertope as an incidence geometry;
- Chirality in hypertopes;
- Construction of hypertopes from groups.

Incidence system

An *incidence system* $\Gamma := (X, *, t, I)$ is a 4-tuple such that

- X is a set whose elements are called the *elements* of Γ;
- I is a finite set whose elements are called the *types* of Γ;
- $t : X \to I$ is a *type function* , associating to each element $x \in X$ of Γ a type $t(x) \in I$;
- $*$ is a binary relation on X called *incidence*, that is reflexive, symmetric and such that for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

Example [Cube] 8 elements of type 0 (vertices), 12 elements of type 1(edges), 6 elements of type 2 (faces).

The *incidence graph* of Γ is the graph whose vertex set is X and where two vertices are joined if the corresponding elements of Γ are incident.

The *rank* of Γ is the cardinality of I.

Incidence Geometry

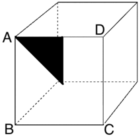
A *flag* is a set of pairwise incident elements of Γ, i.e. a clique of its incidence graph.

The *type* of a flag F is $\{t(x) : x \in F\}$.

Example: $F = \{A, [ABCD]\}$ is a flag of type $\{0, 2\}$.

A *chamber* is a flag of type I.

Example: $\Phi = \{A, [ABCD], [AD]\}$ is a chamber.



An incidence system Γ is a *geometry* or *incidence geometry* if every flag of Γ is contained in a chamber.

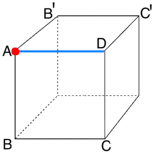
Hypertope

Given a flag F of Γ, the *residue* of F in Γ is the incidence system $\Gamma_F := (X_F, *_F, t_F, I_F)$ where

- X_F all elements not in F that are incident to every element of F;
- $I_F := I \setminus t(F)$;
- t_F and $*_F$ are the restrictions of t and $*$ to X_F and I_F.

An incidence system Γ is *residually connected* when each residue of rank at least two of Γ has a connected incidence graph.

It is called *thin* when every residue of rank one of Γ contains exactly two elements.

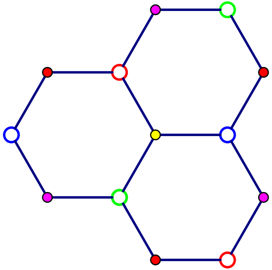


Definition: **A hypertope is a thin residually connected geometry.**

Example of a geometry not thin

Consider the toroidal hypermap $(3,3,3)_{(1,1)}$ that is an embedding of a bipartite graph on torus.

- $\bullet$ - vertex $\circ$ - edge $\#vertices = \#edges = \#faces = 3$



This hypermap has 3 vertices, 3 edges and 3 faces and the incidence graph is $K_{3,3,3}$.

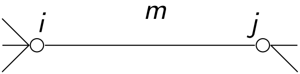
Buekenhout diagram of a hypertope

Lemma: An incidence geometry of rank at least two is thin if and only if all of its rank two residues are thin.

Every thin connected rank 2 geometry is an m-gon for some $m \in \mathbb{N} \cup \{\infty\}$.

Let Γ be a hypertope. The *Buekenhout diagram* $\mathcal{B}(\Gamma)$ of Γ is a graph as follows.

- The vertices are the elements of I;
- There is an edge $\{i, j\}$ with label m whenever $m \neq 2$.



A polytope is a hypertope with linear Buekenhout diagram.

Regular hypertope

Let $\Gamma := (X, *, t, I)$ be an incidence system. A *type preserving automorphism* of Γ is a mapping
$\alpha : (X, I) \to (X, I) : (x, t(x)) \mapsto (\alpha(x), t(\alpha(x))$ where

- α is a bijection on X inducing a bijection on I;
- for each x, $y \in X$, $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- for each x, $t(\alpha(x)) = t(x)$.

Let G be the set of type-preserving automorphisms of Γ is a group denoted by $Aut_I(\Gamma)$.

An incidence system Γ is *flag-transitive* if $Aut_I(\Gamma)$ is transitive on all flags of a given type J for each type $J \subseteq I$.

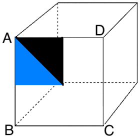
Definition: A *regular hypertope* is a flag-transitive hypertope.

Chiral hypertope

We now extend the notion of chirality in abstract polytopes to incidence geometries.

Let $\Gamma = (X, *, t, I)$ be a thin incidence geometry.

Two chambers Φ and Φ' are *i-adjacent* if they differ only in their elements of type i.



Thanks to thinness there is only one chamber i-adjacent to Φ, that we will denote by Φ^i.

Definition: We say that Γ is *chiral* if $Aut_I(\Gamma)$ has two orbits on the chambers of Γ such that any two adjacent chambers lie in distinct orbits.

C-group

Proposition: Regular hypertopes are C-groups.

A *C-group of rank r* is a pair (G, S) such that G is a group and $S := \{\rho_0, \ldots, \rho_{r-1}\}$ is a generating set of involutions of G that satisfy the following property.

$$\forall I, J \subseteq \{0, \ldots, r-1\}, \langle \rho_i \mid i \in I \rangle \cap \langle \rho_j \mid j \in J \rangle = \langle \rho_k \mid k \in I \cap J \rangle$$

This property is called the *intersection property* and denoted by (IP).

The *Coxeter diagram* $\mathcal{C}(G, S)$ of a C-group (G, S) is a graph satisfying the following.

1. The vertices are the elements of S.
2. Two vertices ρ_i and ρ_j are joined by an edge labelled by $o(\rho_i \rho_j)$ whenever $o(\rho_i \rho_j) \neq 2$.

By convention labels 3 are omitted.

A regular polytope is a *string C-group*, that is, a C-group with linear Coxeter diagram.

Tits algorithm

Let (G, S) with $S := \{\rho_0, \ldots, \rho_{r-1}\}$ be a C-group. Consider the following subgroups.

$$G_i = \langle \rho_j \,|\, j \neq i \rangle$$

For X, $*$ and t defined as follows, $\Gamma := (X, *, t, I)$ is an incidence geometry.

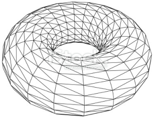
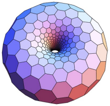
- X the set of all cosets $G_i g$, $g \in G$, $i \in I$;
- $t : X \to I$ defined by $t(G_i g) = i$;
- $G_i g_1 * G_j g_2$ iff $G_i g_1 \cap G_j g_2$ is non-empty in G.

$\Gamma(G; (G_i)_{i \in I})$ is called *coset geometry*.

The subgroups $(G_i)_{i \in I}$ are called the *maximal parabolic subgroups of* Γ.

Toroidal rank 3 hypertopes

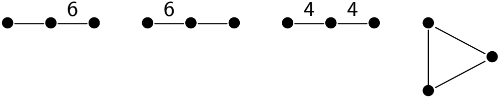
The toroidal polyhedra (rank 3 polytopes) are the toroidal maps, that is, tessellations of the torus by triangles, hexagons and squares.


The group of a toroidal map are quotients of Coxeter groups by groups of translations defined by a vector (b, c). The toroidal maps are denoted by

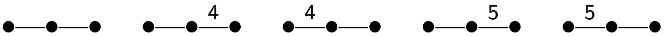
$$\{3,6\}_{(b,c)},\ \{6,3\}_{(b,c)} \text{ and } \{4,4\}_{(b,c)}$$

The other toroidal hypertopes are hypermaps $(3,3,3)_{(b,c)}$ with $(b, c) \neq (1, 1)$ that are hexagonal tessellations of torus.



Diagrams of toroidal hypertopes of rank 4

The *spherical hypertopes* precisely the finite Coxeter groups.
Rank 3 spherical hypertopes are the Platonic solids.



Locally toroidal hypertopes (LTH) of rank 4 are hypertopes whose maximal parabolic subgroups are either a toroidal or spherical hypertopes, with at least one being toroidal.

Nonlinear Coxeter diagrams of rank 4 LTH ($p \in \{3, 4, 5, 6\}$)