Regular and chiral hypertopes

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The genesis of the theory of hypertopes

**Idea:** Extend the theory of polytopes to more general objects by dropping linear condition.

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Thin

Regular polytopes ≡ residually connected
regular geometries
with linear diagram
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A “hypertope” is a generalization of a “polytope” as a “hypermap” is a generalization of a “map”.

- Definition of a hypertope as an incidence geometry;
- Chirality in hypertopes;
- Construction of hypertopes from groups.
Incidence system

An *incidence system* $\Gamma := (X, *, t, I)$ is a 4-tuple such that

- $X$ is a set whose elements are called the *elements* of $\Gamma$;
- $I$ is a finite set whose elements are called the *types* of $\Gamma$;
- $t : X \to I$ is a *type function*, associating to each element $x \in X$ of $\Gamma$ a type $t(x) \in I$;
- $*$ is a binary relation on $X$ called *incidence*, that is reflexive, symmetric and such that for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

**Example [Cube]** 8 elements of type 0 (vertices), 12 elements of type 1(edges), 6 elements of type 2 (faces).

The *incidence graph* of $\Gamma$ is the graph whose vertex set is $X$ and where two vertices are joined if the corresponding elements of $\Gamma$ are incident.

The *rank* of $\Gamma$ is the cardinality of $I$. 
Incidence Geometry

A **flag** is a set of pairwise incident elements of $\Gamma$, i.e. a clique of its incidence graph.

The **type** of a flag $F$ is $\{t(x) : x \in F\}$.

**Example:** $F = \{A, [ABCD]\}$ is a flag of type $\{0, 2\}$.

A **chamber** is a flag of type $I$.

**Example:** $\Phi = \{A, [ABCD], [AD]\}$ is a chamber.

An incidence system $\Gamma$ is a **geometry** or **incidence geometry** if every flag of $\Gamma$ is contained in a chamber.
Hypertope

Given a flag $F$ of $\Gamma$, the *residue* of $F$ in $\Gamma$ is the incidence system $\Gamma_F := (X_F, \star_F, t_F, I_F)$ where

- $X_F$ all elements not in $F$ that are incident to every element of $F$;
- $I_F := I \setminus t(F)$;
- $t_F$ and $\star_F$ are the restrictions of $t$ and $\star$ to $X_F$ and $I_F$.

An incidence system $\Gamma$ is *residually connected* when each residue of rank at least two of $\Gamma$ has a connected incidence graph.

It is called *thin* when every residue of rank one of $\Gamma$ contains exactly two elements.

Definition: A **hypertope** is a thin residually connected geometry.
What is a hypertope?

Example of a geometry not thin

Consider the toroidal hypermap $(3, 3, 3)_{(1,1)}$ that is an embedding of a bipartite graph on torus.

- vertex ○ - edge

$\#v = \#e = \#f = 3$

This hypermap has 3 vertices, 3 edges and 3 faces and the incidence graph is $K_{3,3,3}$. 
**Lemma:** An incidence geometry of rank at least two is thin if and only if all of its rank two residues are thin.

Every thin connected rank 2 geometry is an $m$-gon for some $m \in \mathbb{N} \cup \{\infty\}$.

Let $\Gamma$ be a hypertope. The *Buekenhout diagram* $B(\Gamma)$ of $\Gamma$ is a graph as follows.

- The vertices are the elements of $I$;
- There is an edge $\{i, j\}$ with label $m$ whenever $m \neq 2$.

![Buekenhout diagram](image)

A polytope is a hypertope with linear Buekenhout diagram.
Regular hypertope

Let $\Gamma := (X, *, t, I)$ be an incidence system. A \textit{type preserving automorphism} of $\Gamma$ is a mapping $\alpha : (X, I) \rightarrow (X, I) : (x, t(x)) \mapsto (\alpha(x), t(\alpha(x))$ where

- $\alpha$ is a bijection on $X$ inducing a bijection on $I$;
- for each $x, y \in X$, $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- for each $x$, $t(\alpha(x)) = t(x)$.

Let $G$ be the set of type-preserving automorphisms of $\Gamma$ is a group denoted by $\text{Aut}_I(\Gamma)$.

An incidence system $\Gamma$ is \textit{flag-transitive} if $\text{Aut}_I(\Gamma)$ is transitive on all flags of a given type $J$ for each type $J \subseteq I$.

\textbf{Definition:} A \textit{regular hypertope} is a flag-transitive hypertope.
Chiral hypertope

We now extend the notion of chirality in abstract polytopes to incidence geometries.

Let $\Gamma = (X, *, t, I)$ be a thin incidence geometry.

Two chambers $\Phi$ and $\Phi'$ are \textit{i-adjacent} if they differ only in their elements of type $i$.

Thanks to thinness there is only one chamber $i$-adjacent to $\Phi$, that we will denote by $\Phi'$.

\textbf{Definition:} We say that $\Gamma$ is \textit{chiral} if $Aut_i(\Gamma)$ has two orbits on the chambers of $\Gamma$ such that any two adjacent chambers lie in distinct orbits.
**C-group**

**Proposition:** Regular hypertopes are C-groups.

A **C-group of rank** $r$ is a pair $(G, S)$ such that $G$ is a group and $S := \{\rho_0, \ldots, \rho_{r-1}\}$ is a generating set of involutions of $G$ that satisfy the following property.

$$\forall I, J \subseteq \{0, \ldots, r - 1\}, \langle \rho_i \mid i \in I \rangle \cap \langle \rho_j \mid j \in J \rangle = \langle \rho_k \mid k \in I \cap J \rangle$$

This property is called the *intersection property* and denoted by $(IP)$. The **Coxeter diagram** $\mathcal{C}(G, S)$ of a C-group $(G, S)$ is a graph satisfying the following.

1. The vertices are the elements of $S$.
2. Two vertices $\rho_i$ and $\rho_j$ are joined by an edge labelled by $o(\rho_i \rho_j)$ whenever $o(\rho_i \rho_j) \neq 2$.

By convention labels 3 are omitted.

A regular polytope is a **string C-group**, that is, a C-group with linear Coxeter diagram.
Tits algorithm

Let \((G, S)\) with \(S := \{\rho_0, \ldots, \rho_{r-1}\}\) be a \(C\)-group. Consider the following subgroups.

\[ G_i = \langle \rho_j \mid j \neq i \rangle \]

For \(X, \ast\) and \(t\) defined as follows, \(\Gamma := (X, \ast, t, I)\) is an incidence geometry.

- \(X\) the set of all cosets \(G_i g, \ g \in G, \ i \in I;\)
- \(t : X \rightarrow I\) defined by \(t(G_i g) = i;\)
- \(G_i g_1 \ast G_j g_2\) iff \(G_i g_1 \cap G_j g_2\) is non-empty in \(G.\)

\(\Gamma(G; (G_i)_{i \in I})\) is called coset geometry.

The subgroups \((G_i)_{i \in I}\) are called the maximal parabolic subgroups of \(\Gamma.\)
Toroidal rank 3 hypertopes

The toroidal polyhedra (rank 3 polytopes) are the toroidal maps, that is, tessellations of the torus by triangles, hexagons and squares.

The group of a toroidal map are quotients of Coxeter groups by groups of translations defined by a vector \((b, c)\). The toroidal maps are denoted by \(\{3, 6\}_{(b, c)}\), \(\{6, 3\}_{(b, c)}\) and \(\{4, 4\}_{(b, c)}\).

The other toroidal hypertopes are hypermaps \((3, 3, 3)_{(b, c)}\) with \((b, c) \neq (1, 1)\) that are hexagonal tessellations of torus.

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet & \text{6} & \bullet & \bullet & \text{6} & \bullet & \bullet & \text{4} & \bullet & \bullet & \text{4} & \bullet & \bullet & \triangle & \bullet
\end{array}
\]
The *spherical hypertopes* precisely the finite Coxeter groups. Rank 3 spherical hypertopes are the Platonic solids.

Locally toroidal hypertopes (LTH) of rank 4 are hypertopes whose maximal parabolic subgroups are either a toroidal or spherical hypertopes, with at least one being toroidal.

Nonlinear Coxeter diagrams of rank 4 LTH ($p \in \{3, 4, 5, 6\}$)