$PL(7, 2)$ and the Golomb-Welch Conjecture

5^{th} Combinatorics Day
University of Beira Interior

April 17, 2015
Covilhã

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    1.2. *n*-Crosses
    1.3. *Lee*-Spheres
2. Perfect error correcting Lee codes
    2.1. Perfect error correcting Lee codes
3. Golomb-Welch Conjecture
4. Known results on $PL(n,2)$ codes
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    5.2 $3 \leq |\mathcal{G}_i| \leq 7$
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Tiling R^n by unit cubes

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- $\bigcup_{i \in I} T_i = R^n$
- $\text{int}(T_i) \bigcap \text{int}(T_j) = \emptyset$ for all $i \neq j$.

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Two unit n-cubes are **twins** if they share a complete $n-1$ face

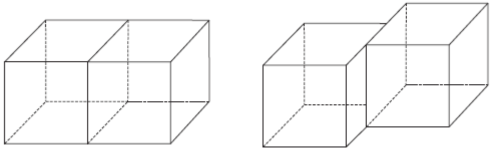

Figure: Twins and no twins.

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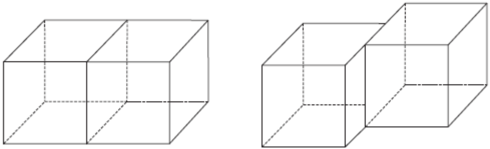

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Lattice Tiling R^n by unit cubes

Lattice: **L** is a **lattice** in R^n of dimension **k** if.

$$\mathbf{L} = \{m_1 v_1 + m_2 v_2 + ... + m_k v_k,\ \ m_i \in \mathbb{Z},\ i = 1, .., k\}$$

with $v_1, v_2, ..., v_k$ linearly independent.

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Being **L** a lattice in R^n of **dimension n**, the set

$$F = \{x_1 v_1 + x_2 v_2 + \ldots + x_n v_n,\ \ 0 \leq x_i \leq 1,\ i = 1,..,n\}$$

is called a **fundamental parallelepiped** of the lattice.

The **volume** of F is independent of the chosen basis and is called the **determinant of L**.

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1907: Minkowski settled the case $\mathbf{n} = \mathbf{3}$ *Diophantische Approximationen* p.67-74.

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1941: **Hajós** settled the Minkowski's Conjecture proving an equivalent conjecture about finite abelian groups.

Hajós's version of the Minkowski's Conjecture (1896):
let G be a finite abelian group.

If $a_1, a_2, ..., a_n$ are elements of G and $r_1, r_2, ..., r_n$ are positive integers such that each element of G is uniquely expressed in the form:

$$a_1^{x_1} ... a_n^{x_n}, \ 0 \le x_1 \ \le r_1 - 1, \ ... \ 0 \le x_n \ \le r_n - 1,$$

then $a_i^{r_i} = e$ for some $i \in \{1, 2, ..., n\}$.

Tilings by crosses

n-cross: Cluster consisting of $2n+1$ unit cubes; a central cube where at each facet another unit cube is attached.

Figure: A 2-cross and a 3-cross.

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Szabó (1981): If $2n+1$ is not a prime, then there exists a $\mathbb{Q}$-tiling of $\mathbb{R}^n$ by crosses that is neither a $\mathbb{Z}$-tiling nor a lattice tiling.

Lee spheres

By a Lee sphere centered at $W = (w_1, \dots, w_n) \in \mathbb{Z}^n$ of radius r, denoted by $S(W, r)$ we mean the set

$$S(W,r) = \{V = (v_1, \dots, v_n) \in \mathbb{Z}^n : \rho_L(V, W) = \sum_{i=1}^{n} |w_i - v_i| \leq r\}.$$

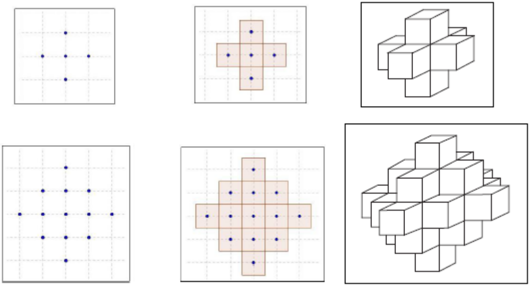

Figure: Lee sphere of radius 1 and 2 in $\mathbb{Z}^2$ and the corresponding unit cube clusters in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

Perfect error correcting Lee codes

Let us consider the metric space $(\mathbb{Z}^n, \rho_L)$. Any subset $\mathcal{M}$ of $\mathbb{Z}^n$, $|\mathcal{M}| \geq 2$, is called a code. The elements of $\mathbb{Z}^n$ will be referred as words and in particular, the elements of $\mathcal{M}$ will be called codewords.

A code $\mathcal{M}$ is a r-error correcting Lee code if

(i) $\forall_{W,V \in \mathcal{M}},\ S(W,r) \cap S(V,r) = \emptyset.$

If, in adition,

(ii) $\cup_{W \in \mathcal{M}} S(W,r) = \mathbb{Z}^n,$

then $\mathcal{M}$ is a perfect r-error correcting Lee code of word length n over $\mathbb{Z}$, shortly a $PL(n,r)$ code.

The elements $V \in S(W,r)$, with $W \in \mathcal{M}$ are said to be **covered** by W.

A $PL(n,r)$ code is a tiling of $\mathbb{Z}^n$ by Lee spheres of radius r

inducing a tiling of $\mathbb{R}^n$ by the corresponding cluster of unit cubes.

$PL(2,r)$ codes

For each $r \geq 2$ there is a tiling of $\mathbb{R}^2$ by clusters of unit 2-cubes associated to Lee Spheres of radius r.

$PL(2,r)$ codes do exist for any $r \geq 2$.

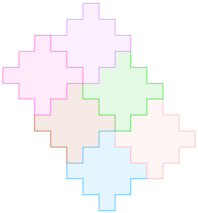

Figure: Tiling of $\mathbb{R}^2$ by Lee Spheres of radius 2.

Golomb-Welch Conjecture

Golomb-Welch Conj. (1969): For any $n \geq 3$ there is no tilings of $\mathbb{R}^n$ by clusters of unit cubes associated with Lee spheres of radius $r \geq 2$.

Equivalently, there are no $PL(n,r)$ codes for $n \geq 3$ and $r \geq 2$.

Gravier, Molard, Payan (1998): There are no $PL(3,r)$ codes for any $r \geq 2$.

Spacapan (2007): There are no $PL(4,r)$ codes for any $r \geq 2$.

P. Horak (2009): There are no $PL(5,r)$ codes for any $r \geq 2$;.
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There are no $PL(6,r)$ codes for any $r \geq 2$.

O. Grosek & P. Horak (2014): There are no **lattice** tilings of $\mathbb{R}^n$ for $7 \leq n \leq 12$ by Lee spheres of radius 2.

What about $PL(7,2)$ codes?

$PL(n,2)$ codes

Let $\mathcal{M}$ be a $PL(n,2)$ code and that $O=(0,0,...,0)\in\mathcal{M}$, which means that:

- all words $V\in\mathbb{Z}^n$ such that $\rho_L(V,O)\leq 2$ are covered by O.

Next level of words to be covered $\Rightarrow$ $\mathcal{V}_3=\{V\in\mathbb{Z}^n:\ \rho_L(O,V)=3\}$.

Let $\mathcal{T}=\{W\in\mathcal{M}$ covering the words of $\mathcal{V}_3\}$. Then, $\mathcal{T}\subset\mathcal{V}_{\mathbf{5}}$,

$\mathcal{T}=\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}\cup\mathcal{E}\cup\mathcal{F}\cup\mathcal{G}$ where

- $\mathcal{A}=\{V\in\mathcal{T}: V$ is of type $[\pm\mathbf{5}]\}$;
- $\mathcal{B}=\{V\in\mathcal{T}: V$ is of type$[\pm\mathbf{4},\pm\mathbf{1}]$;
- $\mathcal{C}=\{V\in\mathcal{T}: V$ is of type$[\pm\mathbf{3},\pm\mathbf{2}]$;
- $\mathcal{D}=\{V\in\mathcal{T}: V$ is of type$[\pm\mathbf{3},\pm\mathbf{1}^\mathbf{2}]$;
- $\mathcal{E}=\{V\in\mathcal{T}: V$ is of type$[\pm\mathbf{2}^\mathbf{2},\pm\mathbf{1}]$;
- $\mathcal{F}=\{V\in\mathcal{T}: V$ is of type$[\pm\mathbf{2},\pm\mathbf{1}^\mathbf{3}]$;
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- $\mathcal{C}=\{V\in\mathcal{T}:V$ is of type$[\pm\mathbf{3},\pm\mathbf{2}]$; $\Longrightarrow$ $\mathbf{c}=|\mathcal{C}|$
- $\mathcal{D}=\{V\in\mathcal{T}:V$ is of type$[\pm\mathbf{3},\pm\mathbf{1}^2]$; $\Longrightarrow$ $\mathbf{d}=|\mathcal{D}|$
- $\mathcal{E}=\{V\in\mathcal{T}:V$ is of type$[\pm\mathbf{2}^2,\pm\mathbf{1}]$; $\Longrightarrow$ $\mathbf{e}=|\mathcal{E}|$
- $\mathcal{F}=\{V\in\mathcal{T}:V$ is of type$[\pm\mathbf{2},\pm\mathbf{1}^3]$; $\Longrightarrow$ $\mathbf{f}=|\mathcal{F}|$
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- All words $V \in \mathbb{Z}^n$ such that $\mu_L(V, O) \leq 2$ are covered by O.

Next level of words to be covered $\Rightarrow \mathcal{V}_3 = \{V \in \mathbb{Z}^n : \ \rho_L(O, V) = 3\}$.

$$\mathcal{V}_3 = \{[\pm \mathbf{3}]; \ [\pm \mathbf{2}, \pm \mathbf{1}]; \ [\pm \mathbf{1}^{\mathbf{3}}]\}.$$

- Each word of type $[\pm \mathbf{3}]$ is covered by one and only one codeword in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ and $|[\pm 3]| = 2n$. Thus,
$$a + b + c + d = 2n$$
- Each word of type $[\pm \mathbf{2}, \pm \mathbf{1}]$ is covered by one codeword in $\mathcal{B}$; 2 codewords in $\mathcal{C}$; 2 codewords in $\mathcal{D}$; 4 codewords in $\mathcal{E}$ and 3 codewords in $\mathcal{F}$. Besides, $|[\pm 2, \pm 1]| = 8\binom{n}{2}$. Thus,
$$b + 2c + 2d + 4e + 3f = 8\binom{n}{2}$$
- Each word of type $[\pm \mathbf{1}^{\mathbf{3}}]$ is covered by one codeword of $\mathcal{D}$; one codeword of $\mathcal{E}$; 4 codewords of $\mathcal{F}$ and 10 words of $\mathcal{G}$. Besides, $|[\pm 1^3]| = 8\binom{n}{3}$. Thus,
$$d + e + 4f + 10g = 8\binom{n}{3}.$$

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Set of signed coordinates $\mathcal{I} = \{+1, +2, ..., +n, -1, -2, ..., -n\}$

Given $\mathcal{H} \subset \mathbb{Z}^n$ denote by

- $\mathcal{H}_i = \{W \in \mathcal{H} \ : \ iw_{|i|} > 0\},\ i \in \mathcal{I}$
- $\mathcal{H}_{ij} = \{W \in \mathcal{H} \ : \ iw_{|i|} > 0 \wedge jw_{|j|} > 0\},\ i,j \in \mathcal{I}\ i \neq j,\ i \neq -j$
- $\mathcal{H}_i^{(k)} = \{W \in \mathcal{H} \ : \ iw_{|i|} > 0 \wedge |w_{|i|}| = k\},\ i \in \mathcal{I},\ k \in \mathbb{Z}^+.$

Relation between the cardinality of each set of codewords and their index subsets can be derived:

- $g = |\mathcal{G}| = \frac{1}{5}\sum_{i\in\mathcal{I}}|\mathcal{G}_i|;\ |\mathcal{G}_i| = \frac{1}{4}\sum_{j\in\mathcal{I}\setminus\{i,-i\}}|\mathcal{G}_{ij};\ |\mathcal{G}_{ij}| = \frac{1}{3}\sum_{k\in\mathcal{I}\setminus\{i,-i,j,-j\}}|\mathcal{G}_{ijk}|.$

Similar equalities for the other subsets of $\mathcal{T}$ can be derived.

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Proposition:

- $|\mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 1,$ *for each* $i \in \mathcal{I}$*;*
- $|\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}| + |\mathcal{C}_i \cap \mathcal{C}_j| + |\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}| + |\mathcal{E}_i^{(2)} \cap \mathcal{E}_j| + |\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}| = 1,$ *for each* $i, j \in \mathcal{I}$, *with* $|i| \neq |j|$
- $|\mathcal{D}_{ijk} \cup \mathcal{E}_{ijk} \cup \mathcal{F}_{ijk} \cup \mathcal{G}_{ijk}| = 1,$ *for each* $i, j, k \in \mathcal{I}$, *with* $|i|$, $|j|$ *and* $|k|$ *distinct between them.*
- $\forall_{i \in \mathcal{I}}, |\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| + 2|\mathcal{D}_i^{(3)} \cup \mathcal{E}_i^{(2)}| + 3|\mathcal{F}_i^{(2)}| = 2(n-1)$ *and*

  $|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(1)} \cup \mathcal{E}_i^{(2)} \cup \mathcal{F}_i^{(1)}| + 2|\mathcal{E}_i^{(1)}| = 2(n-1).$

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Proof: Let V be a word of type $[\pm 2, \pm 1]$, satisfying $iv_{|i|}, jv_{|j|} > 0$, $|v_{|i|}| = 2$ and $|v_{|j|}| = 1$.

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$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Proposition:

$|\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}| + |\mathcal{C}_i \cap \mathcal{C}_j| + |\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}| + |\mathcal{E}_i^{(2)} \cap \mathcal{E}_j| + |\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}| = 1$, for each $i, j \in \mathcal{I}$, with $|i| \neq |j|$

Proof: Let V be a word of type $[\pm 2, \pm 1]$, satisfying $iv_{|i|}, jv_{|j|} > 0$, $|v_{|i|}| = 2$ and $|v_{|j|}| = 1$.

V must be covered by a codeword $W \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F}$ satisfying one and only one of the following conditions:

$W \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}$; $W \in \mathcal{C}_i \cap \mathcal{C}_j$; $W \in \mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}$; $W \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_j$; $W \in \mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}$.

$|\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}| + |\mathcal{C}_i \cap \mathcal{C}_j| + |\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}| + |\mathcal{E}_i^{(2)} \cap \mathcal{E}_j| + |\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}| = 1.$

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Proposition:

- $|D_i \cup E_i| + 3|F_i| + 6|G_i| = 4\binom{n-1}{2}$,
- $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| + 2|\mathcal{F}_{ij}| + 3|\mathcal{G}_{ij}| = 2(n-2); \quad |\mathcal{D}_i \cup \mathcal{E}_i| \leq 2n-1$
- $|G_i| > \frac{|D_i \cup E_i| + (n-1)(n-6)}{3} - \frac{1}{6},\ i \in \mathcal{I};$
- $|\mathcal{F}_i| \leq \frac{8(n-1)+1}{3} - |\mathcal{D}_i \cup \mathcal{E}_i| - \frac{2}{3}|\mathcal{E}_i|,\ i \in \mathcal{I};$
- *If* $n \equiv 1(mod 3)$, *then* $|G_i| \leq \frac{(n-1)(2n-5)}{6}$ *for each* $i \in \mathcal{I}$.
- *If* $n \equiv 0(mod 3)$, *then* $|G_i| \leq \frac{(n-1)(n-3)}{3}$ *for each* $i \in \mathcal{I}$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1]; \ \mathcal{F} := [\pm 2, \pm 1^3]; \ \mathcal{G} := [\pm 1^5]$

Corollary:

- *If* $n \equiv 0 \ (mod\ 3)$, *then* $g \leq \frac{2n(n-1)(n-3)}{15}$;
- *If* $n \equiv 1 \ (mod\ 3)$, *then* $g \leq \frac{n(n-1)(2n-5)}{15}$.

$\mathcal{A} := [\pm 5]$; $\mathcal{B} := [\pm 4, \pm 1]$; $\mathcal{C} := [\pm 3, \pm 2]$; $\mathcal{D} := [\pm 3, \pm 1^2]$; $\mathcal{E} := [\pm 2^2, \pm 1]$; $\mathcal{F} := [\pm 2, \pm 1^3]$; $\mathcal{G} := [\pm 1^5]$

- “Good Solutions" of $\begin{cases} a+b+c+d = 14 \\ b+2c+2d+4e+3f = 168 \\ d+e+4f+10g = 280. \end{cases}$

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1]; \ \mathcal{F} := [\pm 2, \pm 1^3]; \ \mathcal{G} := [\pm 1^5]$

Proposition:

- $|D_i \cup E_i| + 3|F_i| + 6|G_i| = 4\binom{n-1}{2}$,
- $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| + 2|\mathcal{F}_{ij}| + 3|\mathcal{G}_{ij}| = 2(n-2); \quad |\mathcal{D}_i \cup \mathcal{E}_i| \leq 2n-1$
- $|G_i| > \frac{|D_i \cup E_i| + (n-1)(n-6)}{3} - \frac{1}{6}, \ i \in \mathcal{I};$
- $|\mathcal{F}_i| \leq \frac{8(n-1)+1}{3} - |\mathcal{D}_i \cup \mathcal{E}_i| - \frac{2}{3}|\mathcal{E}_i|, \ i \in \mathcal{I};$
- *If* $n \equiv 1 (mod 3)$, *then* $|G_i| \leq \frac{(n-1)(2n-5)}{6}$ *for each* $i \in \mathcal{I}$.
- *If* $n \equiv 1 \ (mod\ 3)$, *then* $g \leq \frac{n(n-1)(2n-5)}{15}$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Proposition:

- $|D_i \cup E_i| + 3|F_i| + 6|G_i| = 60$,
- $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| + 2|\mathcal{F}_{ij}| + 3|\mathcal{G}_{ij}| = 10$; $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}|$ *and* $|\mathcal{G}_{ij}$ *have the same parity,* $|\mathcal{D}_i \cup \mathcal{E}_i| \le 13$
- $3 \le |G_i| \le 9$ *for each* $i \in \mathcal{I}$.
- $|\mathcal{F}_i| \le \frac{49}{3} - |\mathcal{D}_i \cup \mathcal{E}_i| - \frac{2}{3}|\mathcal{E}_i|,\ i \in \mathcal{I}$;
- $6 \le g \le 25$.
- $\forall_{i \in \mathcal{I}}, |\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| + 2|\mathcal{D}_i^{(3)} \cup \mathcal{E}_i^{(2)}| + 3|\mathcal{F}_i^{(2)}| = 12$ *and* $|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(1)} \cup \mathcal{E}_i^{(2)} \cup \mathcal{F}_i^{(1)}| + 2|\mathcal{E}_i^{(1)}| = 12$.

$\mathcal{A} := [\pm 5]$; $\mathcal{B} := [\pm 4, \pm 1]$; $\mathcal{C} := [\pm 3, \pm 2]$; $\mathcal{D} := [\pm 3, \pm 1^2]$; $\mathcal{E} := [\pm 2^2, \pm 1]$; $\mathcal{F} := [\pm 2, \pm 1^3]$; $\mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in \mathcal{I}$, $3 \leq |\mathcal{G}_i| \leq 8$; $9 \leq g \leq 22$.

Proof: By contradiction, assume $|\mathcal{G}_i| = 9$ for some $i \in \mathcal{I}$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1]; \ \mathcal{F} := [\pm 2, \pm 1^3]; \ \mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in I$, $3 \le |\mathcal{G}_i| \le 8$; $9 \le g \le 22$.

Proof: By contradiction, assume $|\mathcal{G}_i| = 9$ for some $i \in I$.

▷ $|\mathcal{G}_i| = \frac{1}{4} \sum_{j \in I \setminus \{i,-i\}} |\mathcal{G}_{ij}| \implies \sum_{j \in I \setminus \{i,-i\}} |\mathcal{G}_{ij}| = 36.$

▷ $|\mathcal{G}_{ij}| = 3$ for any $j \in I \setminus \{i,-i\}$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1]; \ \mathcal{F} := [\pm 2, \pm 1^3]; \ \mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in \mathcal{I}$, $3 \leq |\mathcal{G}_i| \leq 8$; $\ 9 \leq g \leq 22$.

Proof: By contradiction, assume $|\mathcal{G}_i| = 9$ for some $i \in \mathcal{I}$.

▷ $|\mathcal{G}_i| = \frac{1}{4} \sum_{j \in \mathcal{I} \setminus \{i,-i\}} |\mathcal{G}_{ij}| \implies \sum_{j \in \mathcal{I} \setminus \{i,-i\}} |\mathcal{G}_{ij}| = 36.$

▷ $|\mathcal{G}_{ij}| = 3$ for any $j \in \mathcal{I} \setminus \{i,-i\}$.

Let $W_1 \in \mathcal{G}_{i\alpha\beta\gamma\delta}$, $\ \alpha, \beta, \gamma, \delta \in \mathcal{I} \setminus \{i,-i\}$ and $|\alpha|$, $|\beta|$, $|\gamma|$, $|\delta|$ pairwise distinct.

▷ For any $W \in \mathcal{G}_i \setminus \{W_1\}$ there exists, at most, one element $\varepsilon \in \{\alpha, \beta, \gamma, \delta\}$ so that $W \in \mathcal{G}_{i\varepsilon}$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in \mathcal{I}$, $3 \leq |\mathcal{G}_i| \leq 8$; $9 \leq g \leq 22$.

Proof: By contradiction, assume $|\mathcal{G}_i| = 9$ for some $i \in \mathcal{I}$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in \mathcal{I}$, $3 \leq |\mathcal{G}_i| \leq 8$; $9 \leq g \leq 22$.

Proof: By contradiction, assume $|\mathcal{G}_i| = 9$ for some $i \in \mathcal{I}$.

▷ For any $W \in \mathcal{G}_i \setminus \{W_1\}$ there exists, at most, one element $\varepsilon \in \{\alpha, \beta, \gamma, \delta\}$ so that $W \in \mathcal{G}_{i\varepsilon}$.

W_1	i	α	β	γ	δ
W_2	i	α			
W_3	i	α			
W_4	i	β			
W_5	i	β			
W_6	i	γ			
W_7	i	γ			
W_8	i	δ			
W_9	i	δ			

Partial index distribution of the codewords of $\mathcal{G}_i$

$\mathcal{A} := [\pm 5]$; $\mathcal{B} := [\pm 4, \pm 1]$; $\mathcal{C} := [\pm 3, \pm 2]$; $\mathcal{D} := [\pm 3, \pm 1^2]$; $\mathcal{E} := [\pm 2^2, \pm 1]$; $\mathcal{F} := [\pm 2, \pm 1^3]$; $\mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in \mathcal{I}$, $3 \leq |\mathcal{G}_i| \leq 8$; $9 \leq g \leq 22$.

Proof:

Let $\mathcal{J} = \{\alpha, \beta, \gamma, \delta\}$, $\mathcal{J}^- = \{-\alpha, -\beta, -\gamma, -\delta\}$, $\mathcal{K} = \mathcal{I} \setminus (\{i, -i\} \cup \mathcal{J} \cup \mathcal{J}^-) = \{x, -x, y, -y\}$.

W_1	i	α	β	γ	δ
W_2	i	α	k_1	k_2	
W_3	i	α	k_3		
W_4	i	β	k_4	k_5	
W_5	i	β	k_6		
W_6	i	γ	k_7	k_8	
W_7	i	γ	k_9		
W_8	i	δ	k_{10}	k_{11}	
W_9	i	δ	k_{12}		

For each $\varepsilon \in \mathcal{J}$ and $W, W' \in \mathcal{G}_{i\varepsilon} \setminus \{W_1\}$, there are, at least, three distinct elements $k, k', k'' \in \mathcal{K}$ so that $W \in \mathcal{G}_{i\varepsilon kk'}$ and $W' \in \mathcal{G}_{i\varepsilon k''}$.

Figure: Partial index distribution of the codewords of $\mathcal{G}_i$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in I$, $3 \leq |\mathcal{G}_i| \leq 8$; $9 \leq g \leq 22$.

Proof:

Let $\mathcal{J} = \{\alpha, \beta, \gamma, \delta\}$, $\mathcal{J}^- = \{-\alpha, -\beta, -\gamma, -\delta\}$, $\mathcal{K} = I \setminus (\{i, -i\} \cup \mathcal{J} \cup \mathcal{J}^-) = \{x, -x, y, -y\}$.

W_1	i	α	β	γ	δ
W_2	i	α	k_1	k_2	
W_3	i	α	k_3		
W_4	i	β	k_4	k_5	
W_5	i	β	k_6		
W_6	i	γ	k_7	k_8	
W_7	i	γ	k_9		
W_8	i	δ	k_{10}	k_{11}	
W_9	i	δ	k_{12}		

W_1	i	α	β	γ	δ
W_2	i	α	x	y	j_1
W_3	i	α	$-x$	j_2	j_3
W_4	i	β	$-x$	y	j_4
W_5	i	β	k_6		
W_6	i	γ	k_7	k_8	
W_7	i	γ	k_9		
W_8	i	δ	k_{10}	k_{11}	
W_9	i	δ	k_{12}		

$j_1, j_2, j_3 \in \mathcal{J}^-$
$j_4 \neq j_1, j_2, j_3,$

Figure: Partial index distribution of the codewords of $\mathcal{G}_i$.

$\mathcal{A} := [\pm 5];\ \mathcal{B} := [\pm 4, \pm 1];\ \mathcal{C} := [\pm 3, \pm 2];\ \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in I$, $3 \leq |\mathcal{G}_i| \leq 8$; $9 \leq g \leq 22$.

Proof:

Let $\mathcal{J} = \{\alpha, \beta, \gamma, \delta\}$, $\mathcal{J}^- = \{-\alpha, -\beta, -\gamma, -\delta\}$, $\mathcal{K} = I \setminus (\{i, -i\} \cup \mathcal{J} \cup \mathcal{J}^-) = \{x, -x, y, -y\}$.

W_1	i	α	β	γ	δ
W_2	i	α	x	y	j_1
W_3	i	α	$-x$	j_2	j_3
W_4	i	β	$-x$	y	j_4
W_5	i	β	k_6		
W_6	i	γ	k_7	k_8	
W_7	i	γ	k_9		
W_8	i	δ	k_{10}	k_{11}	
W_9	i	δ	k_{12}		

$j_1, j_2, j_3 \in \mathcal{J}^-$
$j_4 \neq j_1, j_2, j_3,$

$y \neq k_6$

If $y \in \{k_7, k_8, k_{10}, k_{11}\}$

$\Downarrow$

$W_2 \in \mathcal{G}_{ixy}$ and $W_4 \in \mathcal{G}_{i,-x,y}$

contradiction

$y = k_9$ or $y = k_{12}$

Figure: Partial index distribution of the codewords of $\mathcal{G}_i$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in \mathcal{I}$, $3 \leq |\mathcal{G}_i| \leq 8$; $9 \leq g \leq 22$.

Proof:

Let $\mathcal{J} = \{\alpha, \beta, \gamma, \delta\}$, $\mathcal{J}^- = \{-\alpha, -\beta, -\gamma, -\delta\}$, $\mathcal{K} = \mathcal{I} \setminus (\{i, -i\} \cup \mathcal{J} \cup \mathcal{J}^-) = \{x, -x, y, -y\}$.

W_1	i	α	β	γ	δ
W_2	i	α	x	y	j_1
W_3	i	α	$-x$	j_2	j_3
W_4	i	β	$-x$	y	j_4
W_5	i	β	k_6		
W_6	i	γ	k_7	k_8	
W_7	i	γ	k_9	$\boxed{j_5}$	$\boxed{j_6}$
W_8	i	δ	k_{10}	k_{11}	
W_9	i	δ	k_{12}		

$j_1, j_2, j_3 \in \mathcal{J}^-$
$j_4 \neq j_1, j_2, j_3,$

$\boxed{y = k_9}$ or $y = k_{12}$

Then $W_7 \in \mathcal{G}_{i\gamma y j_5 j_6}$ $\quad j_5, j_6 \neq j_1, j_4$

$\Downarrow$

$j_5 = j_2$ and $j_6 = j_3$.

contradiction

Figure: Partial index distribution of the codewords of $\mathcal{G}_i$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1]; \ \mathcal{F} := [\pm 2, \pm 1^3]; \ \mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in \mathcal{I}$, $3 \leq |\mathcal{G}_i| \leq 8$; $9 \leq g \leq 22$.

Proof:

Let $\mathcal{J} = \{\alpha, \beta, \gamma, \delta\}$, $\mathcal{J}^- = \{-\alpha, -\beta, -\gamma, -\delta\}$, $\mathcal{K} = \mathcal{I} \setminus (\{i, -i\} \cup \mathcal{J} \cup \mathcal{J}^-) = \{x, -x, y, -y\}$.

W_1	i	α	β	γ	δ
W_2	i	α	x	y	j_1
W_3	i	α	$-x$	j_2	j_3
W_4	i	β	$-x$	y	j_4
W_5	i	β	k_6		
W_6	i	γ	k_7	k_8	
W_7	i	γ	k_9	$\boxed{j_5}$	$\boxed{j_6}$
W_8	i	δ	k_{10}	k_{11}	
W_9	i	δ	k_{12}		

$j_1, j_2, j_3 \in \mathcal{J}^-$
$j_4 \neq j_1, j_2, j_3,$

$\boxed{y = k_9}$ or $y = k_{12}$

Then $W_7 \in \mathcal{G}_{i\gamma y j_5 j_6}$ $\quad j_5, j_6 \neq j_1, j_4$

$\Downarrow$

$j_5 = j_2$ and $j_6 = j_3$.

contradiction

$y = k_{12} \Rightarrow$ contradiction

Figure: Partial index distribution of the codewords of $\mathcal{G}_i$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Lemma

$\forall i \in I$, $|\mathcal{F}_i^{(2)}| \leq 4$. If $|\mathcal{F}_i^{(2)}| = 4$, then $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_j| = 1$ for all $j \in I \setminus \{i, -i\}$

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1]; \ \mathcal{F} := [\pm 2, \pm 1^3]; \ \mathcal{G} := [\pm 1^5]$

Lemma

$\forall i \in I$, $|\mathcal{F}_i^{(2)}| \leq 4$. If $|\mathcal{F}_i^{(2)}| = 4$, then $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_j| = 1$ for all $j \in I \setminus \{i, -i\}$

Lemma

- $|\mathcal{G}_i| = 3 \implies |\mathcal{A}_i| = 1, \ |\mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{E}_i| = 0, \ |\mathcal{D}_i| = |\mathcal{D}_i^{(1)}| = 3$ *(*$|\mathcal{D}_i^{(3)}| = 0$*);*
  $|\mathcal{F}_i| = 13$ *with* $|\mathcal{F}_i^{(2)}| = 4$ *and* $|\mathcal{F}_i^{(1)}| = 9$.
- $|\mathcal{G}_i| = 4 \implies |\mathcal{D}_i \cup \mathcal{E}_i| = 3$ *and* $|\mathcal{F}_i| = 11$; *or*
  $|\mathcal{D}_i| = 6 = |\mathcal{D}_i^{(1)}|, \ |\mathcal{E}_i| = 0, \ |\mathcal{F}_i| = 10, \ |\mathcal{A}_i| = 1, \ |\mathcal{B}_i \cup \mathcal{C}_i| = 0$,
  $|\mathcal{F}_i^{(2)}| = 4$ *and* $|\mathcal{F}_i^{(1)}| = 6$.
- $|\mathcal{G}_i| = 5 \implies |\mathcal{D}_i \cup \mathcal{E}_i| = 0$ *and* $|\mathcal{F}_i| = 10$ *or* $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$ *and* $|\mathcal{F}_i| = 9$ *or*
  $|\mathcal{D}_i \cup \mathcal{E}_i| = 6, \ |\mathcal{F}_i| = 8$ *and* $|\mathcal{D}_i| \geq 3$ *or*
  $|\mathcal{D}_i| = 9 = |\mathcal{D}_i^{(1)}|, \ |\mathcal{A}_i| = 1, \ |\mathcal{B}_i \cup \mathcal{C}_i| = |\mathcal{E}_i| = 0$ *and* $|\mathcal{F}_i| = 7$.
  with $|\mathcal{F}_i^{(2)}| = 4$ *and* $|\mathcal{F}_i^{(1)}| = 3$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1]; \ \mathcal{F} := [\pm 2, \pm 1^3]; \ \mathcal{G} := [\pm 1^5]$

Conditions satisfied by the index subsets of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F}$ for a specific value of $|\mathcal{G}_i|$.

Lemma

- $|\mathcal{G}_i| = 6 \implies |\mathcal{D}_i \cup \mathcal{E}_i| = 0, \ |\mathcal{F}_i| = 8$ *or* ; $|\mathcal{D}_i \cup \mathcal{E}_i| = 3, \ |\mathcal{F}_i| = 7$; *or*
  $|\mathcal{D}_i \cup \mathcal{E}_i| = 6, \ |\mathcal{F}_i| = 6$ *or* $|\mathcal{D}_i \cup \mathcal{E}_i| = 9, |\mathcal{D}_i| \geq 6, \ |\mathcal{F}_i| = 5$ *or*
  $|\mathcal{A}_i| = 1, \ |\mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{E}_i| = 0, \ |\mathcal{D}_i| = |\mathcal{D}_i^{(1)}| = 12, \ |\mathcal{F}_i| = |\mathcal{F}_i^{(2)}| = 4$
- $|\mathcal{G}_i| = 7 \implies |\mathcal{D}_i \cup \mathcal{E}_i| = 3, \ |\mathcal{F}_i| = 5$ *or* $|\mathcal{D}_i \cup \mathcal{E}_i| = 6, \ |\mathcal{F}_i| = 4$; *or*
  $|\mathcal{D}_i \cup \mathcal{E}_i| = 9, \ |\mathcal{D}_i| \geq 3, \ |\mathcal{F}_i| = 63$ *or*
  $|\mathcal{D}_i \cup \mathcal{E}_i| = 12, |\mathcal{D}_i| \geq 9, \ |\mathcal{F}_i| = 2.$

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in \mathcal{I}$, $3 \leq |\mathcal{G}_i| \leq 7$; $9 \leq g \leq 19$.

Proof: By contradiction assume $\exists_{i \in \mathcal{I}} |\mathcal{G}_i| = 8 \Rightarrow \sum_{j \in \mathcal{I} \setminus \{i,-i\}} |\mathcal{G}_{ij}| = 32$.

Recall that $\forall_{j \in \in \mathcal{I} \setminus \{i,-i\}} |\mathcal{G}_{ij}| \leq 3$.

Denote by $\mathcal{J} = \{j \in \mathcal{I} \setminus \{i,-i\} : |\mathcal{G}_{ij}| = 3\}$ ($\Rightarrow |\mathcal{J}| \geq 8$), by
$\mathcal{L} = \{j \in \mathcal{I} \setminus \{i,-i\} : |\mathcal{G}_{ij}| \leq 2\}$ ($\Rightarrow |\mathcal{L}| \leq 4$) and by
$\mathcal{K} = \{j \in \mathcal{I} \setminus \{i,-i\} : |\mathcal{G}_{ij}| = 2\}$.

- There are, at most, four distinct elements $j \in \mathcal{J}$ such that $-j \in \mathcal{L}$
- There are $x, y \in \mathcal{J}$, distinct between them, so that $-x, -y \in \mathcal{J}$; i. e. $|\mathcal{G}_{ix}| = |\mathcal{G}_{i,-x}| = |\mathcal{G}_{iy}| = |\mathcal{G}_{i,-y}| = 3$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Denote by $\mathcal{J} = \{j \in \mathcal{I} \backslash \{i, -i\} : |\mathcal{G}_{ij}| = 3\}$ ($\Rightarrow$ $|\mathcal{J}| \geq 8$), by
$\mathcal{L} = \{j \in \mathcal{I} \backslash \{i, -i\} : |\mathcal{G}_{ij}| \leq 2\}$ ($\Rightarrow$ $|\mathcal{L}| \leq 4$) and by
$\mathcal{K} = \{j \in \mathcal{I} \backslash \{i, -i\} : |\mathcal{G}_{ij}| = 2\}$.

$$|\mathcal{G}_{ix}| = |\mathcal{G}_{i,-x}| = |\mathcal{G}_{iy}| = |\mathcal{G}_{i,-y}| = 3.$$

Step 1: $|\mathcal{J}| = 8$; $|\mathcal{K}| = |\mathcal{L}| = 4$ and the partial index distribution of the codewords $W_1, \ldots, W_8 \in \mathcal{G}_i$ satisfies:

W_1	i	k_1	x	y
W_2	i	k_2	x	$-y$
W_3	i	k_3	x	
W_4	i	k_4	$-x$	y
W_5	i	k_5	$-x$	$-y$
W_6	i	k_6	$-x$	
W_7	i	k_7	y	
W_8	i	k_8	$-y$	

where $x, -x, y, -y \in \mathcal{J}$,

$x \neq y$ and

$k_1, \ldots, k_8 \in \mathcal{K}$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Denote by $\mathcal{J} = \{j \in \mathcal{I} \backslash \{i, -i\} : |\mathcal{G}_{ij}| = 3\}$ ($\Rightarrow$ $|\mathcal{J}| \geq 8$), and by
$\mathcal{K} = \{j \in \mathcal{I} \backslash \{i, -i\} : |\mathcal{G}_{ij}| = 2\}$.

$$|\mathcal{G}_{ix}| = |\mathcal{G}_{i,-x}| = |\mathcal{G}_{iy}| = |\mathcal{G}_{i,-y}| = 3.$$

Step 2: If $k \in \mathcal{K}$, then $-k \in \mathcal{K}$.

W_1	i	k_1	x	y	α
W_2	i	k_2	x	$-y$	β
W_3	i	k_3	x	γ	δ
W_4	i	$\boxed{k_2}$	$-x$	y	
W_5	i	$\boxed{k_1}$	$-x$	$-y$	
W_6	i	$\boxed{k_3}$	$-x$	α	β
W_7	i	k_7	y	β	
W_8	i	$\boxed{k_7}$	$-y$	α	

$\boxed{\mathcal{K} = \{k_1, k_2, k_3, k_7\}}$

$$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$$
$$\mathcal{E} := [\pm 2^2, \pm 1]; \ \mathcal{F} := [\pm 2, \pm 1^3]; \ \mathcal{G} := [\pm 1^5]$$

Denote by $\mathcal{J} = \{j \in \mathcal{I} \backslash \{i, -i\} : |\mathcal{G}_{ij}| = 3\}$ ($\Rightarrow$ $|\mathcal{J}| \geq 8$), and by
$\mathcal{K} = \{j \in \mathcal{I} \backslash \{i, -i\} : |\mathcal{G}_{ij}| = 2\}$.

$$|\mathcal{G}_{ix}| = |\mathcal{G}_{i,-x}| = |\mathcal{G}_{iy}| = |\mathcal{G}_{i,-y}| = 3.$$

Step 2: If $k \in \mathcal{K}$, then $-k \in \mathcal{K}$.

W_1	i	k_1	x	y	α
W_2	i	k_2	x	$-y$	β
W_3	i	k_3	x	γ	δ
W_4	i	$\boxed{k_2}$	$-x$	y	$\boxed{\gamma}$
W_5	i	$\boxed{k_1}$	$-x$	$-y$	$\boxed{\delta}$
W_6	i	$\boxed{k_3}$	$-x$	α	β
W_7	i	k_7	y	β	$\boxed{\delta}$
W_8	i	$\boxed{k_7}$	$-y$	α	$\boxed{\gamma}$

$\boxed{\mathcal{K} = \{k_1, k_2, k_3, k_7\}}$

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Denote by $\mathcal{J} = \{j \in I \backslash \{i, -i\} : |\mathcal{G}_{ij}| = 3\}$ ($\Rightarrow \ |\mathcal{J}| \geq 8$), and by
$\mathcal{K} = \{j \in I \backslash \{i, -i\} : |\mathcal{G}_{ij}| = 2\}$.

Step 3: $|\mathcal{F}_i| = 0$.

Step 4: $\forall_{j \in \mathcal{J}}\ |\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| = 1;\ \ \forall_{k \in \mathcal{K}}\ |\mathcal{D}_{ik} \cup \mathcal{E}_{ik}| = 4$. Besides, if $k \in \mathcal{K}$ there are codewords $V_1, V_2 \in \mathcal{G}_{ik}$ and $U_1, \ldots, U_4 \in \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$ whose index distribution satisfies:

V_1	i	k	v_1	v_2	v_3
V_2	i	k	v_4	v_5	v_6

$V_1, V_2 \in \mathcal{G}_{ik}$
$\{v_1, \ldots, v_6\} \subset \mathcal{J}$

U_1	i	k	u
U_2	i	k	$-u$
U_3	i	k	j_1
U_4	i	k	j_2

$U_1, \ldots, U_4 \in \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$.
$u, -u \in \mathcal{K} \backslash \{k, -k\}$
$j_1, j_2 \in \mathcal{J},\ j_1 \neq j_2$

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1]; \ \mathcal{F} := [\pm 2, \pm 1^3]; \ \mathcal{G} := [\pm 1^5]$

Denote by $\mathcal{J} = \{j \in I \backslash \{i, -i\} : |\mathcal{G}_{ij}| = 3\}$ ($\Rightarrow \ |\mathcal{J}| \ge 8$), and by
$\mathcal{K} = \{j \in I \backslash \{i, -i\} : |\mathcal{G}_{ij}| = 2\}$.

Step 5: $|\mathcal{G}_i| \ne 8$.
Recall $[\pm 2, \pm 1]$ are coverd by codewords of $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$.

$V_1, V_2 \in \mathcal{G}_{ik}$
$\{v_1, \ldots, v_6\} \subset \mathcal{J}$

V_1	i	k	v_1	v_2	v_3
V_2	i	k	v_4	v_5	v_6

U_1	i	k	u
U_2	i	k	$-u$
U_3	i	k	j_1
U_4	i	k	j_2

$U_1, \ldots, U_4 \in \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$.
$u, -u \in \mathcal{K} \backslash \{k, -k\}$
$j_1, j_2 \in \mathcal{J}, \ j_1 \ne j_2$

$\mathcal{H} \Rightarrow$ set of words of type $[\pm 2, \pm 1]$

$P_1 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_1}^{(1)}$

$P_2 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_2}^{(1)}$

Recall $[\pm 2, \pm 1]$ are coverd by codewords of $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Step 5: $|\mathcal{G}_i| \neq 8$.

	i	k	u	$-u$	j_1	j_2
U_1	x	x	x			
U_2	x	x		x		
U_3	x	x			x	
U_4	x	x				x
P_1	± 2				± 1	
P_2	± 2					± 1

Index distribution of $U_1, \dots, U_4 \in \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$

and index value distribution of P_1 and P_2

- P_1 may only be covered by $U_3 \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}) \cup \mathcal{C}_{ij_1} \cup (\mathcal{D}_i^{(3)} \cap \mathcal{D}_{j_1}^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_1});$
- P_2 may only be covered by $U_4 \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_2}^{(1)}) \cup \mathcal{C}_{ij_2} \cup (\mathcal{D}_i^{(3)} \cap \mathcal{D}_{j_2}^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_2})$

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Step 5: $|\mathcal{G}_i| \neq 8$.

	i	k	u	$-u$	j_1	j_2
U_1	x	x	x			
U_2	x	x		x		
U_3	x	x			x	
U_4	x	x				x
P_1	± 2				± 1	
P_2	± 2					± 1

Index distribution of $U_1, ..., U_4 \in \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$

and index value distribution of P_1 and P_2

- P_1 may only be covered by $U_3 \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}) \cup \mathcal{C}_{ij_1} \cup (\mathcal{D}_i^{(3)} \cap \mathcal{D}_{j_1}^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_1});$
- P_2 may only be covered by $U_4 \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_2}^{(1)}) \cup \mathcal{C}_{ij_2} \cup (\mathcal{D}_i^{(3)} \cap \mathcal{D}_{j_2}^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_2})$

But $|\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}| + |\mathcal{C}_i \cap \mathcal{C}_j| + |\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}| + |\mathcal{E}_i^{(2)} \cap \mathcal{E}_j| + |\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}| = 1$ and

$U_3, U_4 \in (\mathcal{D}_i^{(3)} \cap \mathcal{D}_k^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_k)$ $\Rightarrow$ either P_1 is not covered by U_3 or

P_2 is not covered by U_4.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Step 5: $|\mathcal{G}_i| \neq 8$.

- Assuming WLOG P_1 not covered by $U_3 \Rightarrow P_1$ covered by $V \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}) \cup \mathcal{C}_{ij_1}$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Step 5: $|\mathcal{G}_i| \neq 8$.

- Assuming WLOG P_1 not covered by $U_3 \Rightarrow P_1$ covered by $V \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}) \cup \mathcal{C}_{ij_1}$.

  Both cases $V \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)})$ and $V \in \cup \mathcal{C}_{ij_1}$ lead to a contradiction.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Step 5: $|\mathcal{G}_i| \neq 8$.

- Assuming WLOG P_1 not covered by $U_3 \Rightarrow P_1$ covered by $V \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}) \cup \mathcal{C}_{ij_1}$.

  Both cases $V \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)})$ and $V \in \cup \mathcal{C}_{ij_1}$ lead to a contradiction.

Result: $\forall_{i \in I}\ 3 \le |\mathcal{G}_i| \le 7,\ 9 \le g \le 19.$

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Proposition:

Let $\mathcal{J} = \{i \in \mathcal{I} : |\mathcal{G}_i| = p \wedge |\mathcal{F}_i| = q\}$.

- $p = 3 \Rightarrow q = 13$ *and* $|\mathcal{J}| \leq 5$;
- $p = 4 \Rightarrow q = 10$ *and* $|\mathcal{J}| \leq 3$;
- $p = 5 \Rightarrow$ ($q = 10$ *and* $|\mathcal{J}| \leq 4$) *or* ($q = 7$ *and* $|\mathcal{J}| \leq 2$);
- $p = 6 \Rightarrow$ ($q = 8$ *and* $|\mathcal{J}| \leq 3$) *or* ($q = 7$ *and* $|\mathcal{J}| \leq 8$) *or* ($q = 5$ *and* $|\mathcal{J}| \leq 5$);
- $p = 7 \Rightarrow$ ($q = 5$ *and* $|\mathcal{J}| \leq 5$) *or* ($q = 2$ *and* $|\mathcal{J}| \leq 2$).

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Proposition:

Let $\mathcal{J} = \{i \in \mathcal{I} : |\mathcal{G}_i| = p \wedge |\mathcal{F}_i| = q\}$.

- $p = 3 \Rightarrow q = 13$ *and* $|\mathcal{J}| \leq 5$;
- $p = 4 \Rightarrow q = 10$ *and* $|\mathcal{J}| \leq 3$;
- $p = 5 \Rightarrow$ ($q = 10$ *and* $|\mathcal{J}| \leq 4$) *or* ($q = 7$ *and* $|\mathcal{J}| \leq 2$);
- $p = 6 \Rightarrow$ ($q = 8$ *and* $|\mathcal{J}| \leq 3$) *or* ($q = 7$ *and* $|\mathcal{J}| \leq 8$) *or* ($q = 5$ *and* $|\mathcal{J}| \leq 5$);
- $p = 7 \Rightarrow$ ($q = 5$ *and* $|\mathcal{J}| \leq 5$) *or* ($q = 2$ *and* $|\mathcal{J}| \leq 2$).

Corollary:

$g \neq 9$ *and* $g \neq 10$.

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Proposition:

Let $\mathcal{J} = \{i \in \mathcal{I} : |\mathcal{G}_i| = p \wedge |\mathcal{F}_i| = q\}$.

- $p = 3 \Rightarrow q = 13$ *and* $|\mathcal{J}| \leq 5$;
- $p = 4 \Rightarrow q = 10$ *and* $|\mathcal{J}| \leq 3$;
- $p = 5 \Rightarrow$ ($q = 10$ *and* $|\mathcal{J}| \leq 4$) *or* ($q = 7$ *and* $|\mathcal{J}| \leq 2$);
- $p = 6 \Rightarrow$ ($q = 8$ *and* $|\mathcal{J}| \leq 3$) *or* ($q = 7$ *and* $|\mathcal{J}| \leq 8$) *or* ($q = 5$ *and* $|\mathcal{J}| \leq 5$);
- $p = 7 \Rightarrow$ ($q = 5$ *and* $|\mathcal{J}| \leq 5$) *or* ($q = 2$ *and* $|\mathcal{J}| \leq 2$).

Corollary:

$g \neq 9$ *and* $g \neq 10$.

$g = 9 = \frac{1}{5} \sum_{i \in \mathcal{I}} |\mathcal{G}_i| \Rightarrow \sum_{i \in \mathcal{I}} |\mathcal{G}_i| = 45 = 11 \times 3 + 3 \times 4 \Rightarrow |\mathcal{J}| \geq 11$ (contradiction).

$g = 10 \Rightarrow \sum_{i \in \mathcal{I}} |\mathcal{G}_i| = 50 = 6 \times 3 + 8 \times 4 \Rightarrow |\mathcal{J}| \geq 6$ (contradiction).

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in I$ $|\mathcal{G}_i| \neq 3$. More precisely, $\forall_{i \in I}\ 4 \leq |\mathcal{G}_i| \leq 7,\ 12 \leq g \leq 19.$

Proof: By contradiction assume $\exists_{i \in I} |\mathcal{G}_i| = 3 \Rightarrow \sum_{j \in I \setminus \{i,-i\}} |\mathcal{G}_{ij}| = 12.$

Step 1: $|\mathcal{G}_i| = 3 \Rightarrow |\mathcal{D}_i| = 3, |\mathcal{E}_i| = 0,\ |\mathcal{F}_i| = 13$ and $|\mathcal{J} = \{i \in I : |\mathcal{G}_i| = 3 \wedge |\mathcal{F}_i| = 13\}| \leq 5.$

Step 2: $|\mathcal{G}_i| = 3 \Rightarrow \exists_{\alpha,\beta,\gamma \in I \setminus \{i,-i\}} |\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$, α, β, γ distinct between them;

$\forall_{\omega \in I \setminus \{i,-i,\alpha,\beta,\gamma\}}\ |\mathcal{F}_{i\omega}| \leq 3.$

$\exists_{U_1,U_2,U_3,U_4} \in \mathcal{F}_i$ satisfying

U_1	i	α	β	x_1
U_2	i	α	γ	x_2
U_3	i	β	γ	x_3
U_4	i	y_1	y_2	y_3

$x_1, x_2, x_3, y_1, y_2, y_3 \in \mathcal{I} \setminus \{i, -i, \alpha, \beta, \gamma\}$

$\mathcal{A} := [\pm 5]$; $\mathcal{B} := [\pm 4, \pm 1]$; $\mathcal{C} := [\pm 3, \pm 2]$; $\mathcal{D} := [\pm 3, \pm 1^2]$;
$\mathcal{E} := [\pm 2^2, \pm 1]$; $\mathcal{F} := [\pm 2, \pm 1^3]$; $\mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in \mathcal{I}$ $|\mathcal{G}_i| \neq 3$. More precisely, $\forall_{i\in\mathcal{I}}\ 4 \leq |\mathcal{G}_i| \leq 7,\ 12 \leq g \leq 19.$

Step 3:

$|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5 \Rightarrow |\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = |\mathcal{G}_{i\gamma}| = 0.$

$\exists_{\delta,\varepsilon,\theta\in\mathcal{I}\setminus\{i,-i,\alpha,\beta,\gamma\}}\ |\mathcal{G}_{i\delta}| = |\mathcal{G}_{i\varepsilon}| = |\mathcal{G}_{i\theta}| = 2$

and $\forall_{\omega\in\mathcal{I}\setminus\{i,-i,\alpha,\beta,\gamma,\delta,\varepsilon,\theta\}}\ |\mathcal{G}_{i\omega}| = 1$

The 3 codewords $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfy:

W_1	i	δ	ε	z_1	z_2
W_2	i	δ	θ	z_3	z_4
W_3	i	ε	θ	z_5	z_6

$\delta, \varepsilon, \theta, z_1, z_2, \ldots z_6 \in \mathcal{I}\setminus\{i, -i, \alpha, \beta, \gamma\}$
distinct between them

$$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$$
$$\mathcal{E} := [\pm 2^2, \pm 1]; \ \mathcal{F} := [\pm 2, \pm 1^3]; \ \mathcal{G} := [\pm 1^5]$$

Proposition:

For each $i \in \mathcal{I}$ $|\mathcal{G}_i| \neq 3$. More precisely, $\forall_{i \in \mathcal{I}}$ $4 \leq |\mathcal{G}_i| \leq 7$, $12 \leq g \leq 19$.

Step 4: Characterization of the index distributions of $\mathcal{G}_i \cup \mathcal{F}_i$ having into account their interaction.

W_1	i	δ	ε	z_1	z_2
W_2	i	δ	θ	z_3	z_4
W_3	i	ε	θ	z_5	z_6

$W_1, W_2, W_3 \in \mathcal{G}_i$

$\delta, \varepsilon, \theta, z_1, z_2, \ldots z_6 \in \mathcal{I} \setminus \{i, -i, \alpha, \beta, \gamma\}$

U_1	i	j	k	l
U_2	i	j	γ	x_1
U_3	i	k	γ	x_2
U_4	i	y_1	y_2	y_3

Codewords of $\mathcal{F}_i$

$\delta = -j$ and $\varepsilon = -k$

I) $\delta, \varepsilon, \theta \neq l$;

1.1 $\theta = -j$

W_1	i	δ	ε	l	z_1
W_2	i	δ	$-j$	z_2	z_3
W_3	i	ε	$-j$	z_4	z_5

1.2 $\theta = -l$

W_1	i	δ	ε	l	z_1
W_2	i	δ	$-l$	z_2	z_3
W_3	i	ε	$-l$	z_4	z_5

II) $\exists^1\, \omega \in \{\delta, \varepsilon, \theta\}$ such that $\omega = l$.

2.1

W_1	i	l	$-j$	m	n
W_2	i	l	$-k$	z_1	z_2
W_3	i	$-j$	$-k$	z_3	z_4

2.2

W_1	i	l	$-j$	z_1	z_2
W_2	i	l	m	z_3	z_4
W_3	i	$-j$	m	z_5	z_6

2.3

W_1	i	l	m	z_1	z_2
W_2	i	l	n	z_3	z_4
W_3	i	m	n	z_5	z_6

Each one of these cases are subdivided in several other according with the available possibilities to fill the unknown coordinates.

$$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$$
$$\mathcal{E} := [\pm 2^2, \pm 1]; \ \mathcal{F} := [\pm 2, \pm 1^3]; \ \mathcal{G} := [\pm 1^5]$$

Proposition:

For each $i \in I$ $|\mathcal{G}_i| \neq 3$. *More precisely,* $\forall_{i \in I}$ $4 \leq |\mathcal{G}_i| \leq 7$, $12 \leq g \leq 19$.

Step 4: Characterization of the index distributions of $\mathcal{G}_i \cup \mathcal{F}_i$ having into account their interaction.

I) $\delta, \varepsilon, \theta \neq l$;

1.1

W_1	i	δ	ε	l	z_1
W_2	i	δ	$-j$	z_2	z_3
W_3	i	ε	$-j$	z_4	z_5

1.1.1 $\delta = -k$

W_1	i	$-k$	m	l	n
W_2	i	$-k$	$-j$	z_1	z_2
W_3	i	m	$-j$	z_3	z_4

$|\mathcal{G}_{i,-k}| = 2$

1.1.2.1 $W_1 \in \mathcal{G}_{i,-k}$

W_1	i	m	n	l	$-k$
W_2	i	m	$-j$	z_1	z_2
W_3	i	n	$-j$	z_3	z_4

1.1.2.2 $W_1 \notin \mathcal{G}_{i,-k}$

W_1	i	m	n	l	o
W_2	i	m	$-j$	$-k$	z_1
W_3	i	n	$-j$	z_2	z_3

$|\mathcal{G}_{i,-k}| = 1$

U_1	i	j	k	l
U_2	i	j	γ	x_1
U_3	i	k	γ	x_2
U_4	i	y_1	y_2	y_3

Codewords of $\mathcal{F}_i$

$$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$$
$$\mathcal{E} := [\pm 2^2, \pm 1]; \ \mathcal{F} := [\pm 2, \pm 1^3]; \ \mathcal{G} := [\pm 1^5]$$

Proposition:

For each $i \in \mathcal{I}$ $|\mathcal{G}_i| \neq 3$. *More precisely,* $\forall_{i \in \mathcal{I}}$ $4 \leq |\mathcal{G}_i| \leq 7$, $12 \leq g \leq 19$.

Step 4: Characterization of the index distributions of $\mathcal{G}_i \cup \mathcal{F}_i$ having into account their interaction.

$\mathcal{G}_i$

W_1	i	$-k$	m	l	n
W_2	i	$-k$	$-j$	$-o$	$-n$
W_3	i	m	$-j$	o	$-l$

$\mathcal{F}_{ij}$

U_1	i	j	k	l
U_2	i	j	$-m$	$-n$
U_5	i	j	m	$-o$
U_6	i	j	$-k$	o
U_7	i	j	$-l$	n

$\mathcal{F}_{ik}$

U_3	i	k	$-m$	o
U_8	i	k	m	$-n$
U_9	i	k	$-j$	n
U_{10}	i	k	$-l$	$-o$

$\mathcal{F}_{i\text{-}m}$

U_{11}	i	$-m$	$-j$	l
U_{12}	i	$-m$	$-k$	$-l$
U_{13}	i	$-m$	$-o$	n

U_4	i	l	$-n$	o

Codewords of $\mathcal{F}_i \cup \mathcal{G}_i$

$|\mathcal{G}_{im}| = 2 = |\mathcal{F}_{im}|$ and $3 \leq |\mathcal{G}_m| \leq 7 \Rightarrow |\mathcal{G}_m \backslash \mathcal{G}_i| \geq 1$

$W_1 \in \mathcal{G}_{i,-k,m,l,n}$, $W_3 \in \mathcal{G}_{i,m,-j,-n,-o}$, $U_5 \in \mathcal{F}_{i,j,m,-l}$ and $U_8 \in \mathcal{F}_{ikmo}$

induce a partition $\mathcal{P}$ of $\mathcal{I} \backslash \{i, m, -m\}$:

$$\boxed{\mathcal{P}_1 = \{-k, l, n\}; \ \mathcal{P}_2 = \{-j, -n, -o\}; \ \mathcal{P}_3 = \{j, -l\}; \ \mathcal{P}_4 = \{k, o\}; \ \mathcal{P}_5 = \{-i\}}$$

$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$
$\mathcal{E} := [\pm 2^2, \pm 1]; \ \mathcal{F} := [\pm 2, \pm 1^3]; \ \mathcal{G} := [\pm 1^5]$

Proposition:

For each $i \in \mathcal{I}$ $|\mathcal{G}_i| \neq 3$. More precisely, $\forall_{i \in \mathcal{I}}$ $4 \leq |\mathcal{G}_i| \leq 7$, $12 \leq g \leq 19$.

$W_1 \in \mathcal{G}_{i,-k,m,l,n}$, $\quad W_3 \in \mathcal{G}_{i,m,-j,-n,-o}$, $\quad U_5 \in \mathcal{F}_{i,j,m,-l}$ and $\quad U_8 \in \mathcal{F}_{ikmo}$

$\mathcal{P}$ of $\mathcal{I} \backslash \{i, m, -m\}$: $\boxed{\mathcal{P}_1 = \{-k, l, n\}; \ \mathcal{P}_2 = \{-j, -n, -o\}; \ \mathcal{P}_3 = \{j, -l\}; \ \mathcal{P}_4 = \{k, o\}; \ \mathcal{P}_5 = \{-i\}}$

Possibilities for $W \in \mathcal{G}_m \backslash \mathcal{G}_i$

1	m	$-i$	$-k$	$-n$	o
2				$-n$	$-l$
3				$-o$	j
4				$-o$	$-l$
5				o	j
6			l	$-j$	k
7				$-j$	o
8				$-n$	k
9				$-o$	k
10				$-n$	j
11				$-o$	j
12				o	j

13	m	$-i$	n	$-j$	o
14				$-o$	k
15				$-j$	$-l$
16				$-o$	$-l$
17				k	j
18				k	$-l$
19				o	j
20				o	$-l$
21			$-j$	k	$-l$
22			$-n$	k	j
23				k	$-l$
24				o	j
25				o	$-l$
26			$-o$	k	j

$$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$$
$$\mathcal{E} := [\pm 2^2, \pm 1];\ \mathcal{F} := [\pm 2, \pm 1^3];\ \mathcal{G} := [\pm 1^5]$$

Proposition:

For each $i \in I$ $|\mathcal{G}_i| \neq 3$. *More precisely,* $\forall_{i \in I}$ $4 \leq |\mathcal{G}_i| \leq 7$, $12 \leq g \leq 19$.

Step 5: Identification of a partition of the index set relevant to the index distributions of $\mathcal{G}_i \cup \mathcal{F}_i$.

Possibilities for $M \in \mathcal{F}_m \backslash \mathcal{F}_i$

<table>
<tr><td rowspan="19">m</td><td rowspan="5">$-k$</td><td rowspan="3">$-l$</td><td>j</td></tr>
<tr><td>$-o$</td></tr>
<tr><td>$-n$</td></tr>
<tr><td rowspan="2">$-n$</td><td>o</td></tr>
<tr><td>j</td></tr>
<tr><td rowspan="7">l</td><td rowspan="3">$-j$</td><td>$-o$</td></tr>
<tr><td>k</td></tr>
<tr><td>$-n$</td></tr>
<tr><td rowspan="2">o</td><td>j</td></tr>
<tr><td>k</td></tr>
<tr><td rowspan="2">$-o$</td><td>k</td></tr>
<tr><td>$-n$</td></tr>
<tr><td rowspan="7">n</td><td rowspan="4">k</td><td>o</td></tr>
<tr><td>$-l$</td></tr>
<tr><td>j</td></tr>
<tr><td>$-o$</td></tr>
<tr><td rowspan="2">$-o$</td><td>$-j$</td></tr>
<tr><td>$-l$</td></tr>
<tr><td>o</td><td>j</td></tr>
</table>

<table>
<tr><td rowspan="19">m</td><td rowspan="2">k</td><td rowspan="2">j</td><td>o</td></tr>
<tr><td>$-l$</td></tr>
<tr><td rowspan="3">$-n$</td><td rowspan="2">$-l$</td><td>j</td></tr>
<tr><td>$-o$</td></tr>
<tr><td>l</td><td>j</td></tr>
<tr><td>$-j$</td><td>$-o$</td><td>k</td></tr>
<tr><td>o</td><td>j</td><td>$-n$</td></tr>
<tr><td rowspan="12">$-i$</td><td rowspan="6">$-k$</td><td>$-j$</td></tr>
<tr><td>o</td></tr>
<tr><td>$-l$</td></tr>
<tr><td>j</td></tr>
<tr><td>$-o$</td></tr>
<tr><td>$-n$</td></tr>
<tr><td rowspan="6">l</td><td>$-j$</td></tr>
<tr><td>o</td></tr>
<tr><td>j</td></tr>
<tr><td>$-o$</td></tr>
<tr><td>k</td></tr>
<tr><td>$-n$</td></tr>
</table>

<table>
<tr><td rowspan="20">m</td><td rowspan="20">$-i$</td><td rowspan="6">n</td><td>$-j$</td></tr>
<tr><td>o</td></tr>
<tr><td>$-l$</td></tr>
<tr><td>j</td></tr>
<tr><td>$-o$</td></tr>
<tr><td>k</td></tr>
<tr><td rowspan="3">$-j$</td><td>$-o$</td></tr>
<tr><td>k</td></tr>
<tr><td>$-n$</td></tr>
<tr><td rowspan="3">o</td><td>j</td></tr>
<tr><td>k</td></tr>
<tr><td>$-n$</td></tr>
<tr><td rowspan="4">$-l$</td><td>j</td></tr>
<tr><td>$-o$</td></tr>
<tr><td>k</td></tr>
<tr><td>$-n$</td></tr>
<tr><td rowspan="2">j</td><td>k</td></tr>
<tr><td>$-n$</td></tr>
<tr><td rowspan="2">$-o$</td><td>k</td></tr>
<tr><td>$-n$</td></tr>
</table>

$$\mathcal{A} := [\pm 5]; \mathcal{B} := [\pm 4, \pm 1]; \mathcal{C} := [\pm 3, \pm 2]; \mathcal{D} := [\pm 3, \pm 1^2];$$
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Proposition:

For each $i \in \mathcal{I}$ $|\mathcal{G}_i| \neq 3$. *More precisely,* $\forall_{i \in \mathcal{I}}$ $4 \leq |\mathcal{G}_i| \leq 7$, $12 \leq g \leq 19$.

Step 6: New results about the index distributions of $\mathcal{G} \cup \mathcal{F}$ han«ving into account the known distributions.

- $3 \leq |\mathcal{G}_m| \leq 6$;
- if $|\mathcal{G}_m| = 3$, then $|\mathcal{F}_m| = 13$ with $|\mathcal{F}_m^{(2)}| = 4$;
- if $|\mathcal{G}_m| = 4$ and $|\mathcal{F}_m| = 10$, then $|\mathcal{F}_m^{(2)}| = 4$;
- if $|\mathcal{G}_m| = 5$ and $|\mathcal{F}_m| = 7$, then $|\mathcal{F}_m^{(2)}| = 4$;
- if $|\mathcal{G}_m| = 6$ and $|\mathcal{F}_m| = 4$, then $|\mathcal{F}_m^{(2)}| = 4$;
- If $|\mathcal{F}_m^{(2)}| = 4$, then $\mathcal{F}_m^{(2)} = \{U_8, M, M', M''\}$, where $U_8 \in \mathcal{F}_{i,k,m,-n}$ and M, M', M'' satisfy one of the following conditions:

M	m, j, -k, -l	
M'	m, -o, l, -j	m, -o, n, -j
M''	m, -i, n, o	m, -i, l, o

M	m j l o		
M'	m, -o, -k, -l	m, -o, n, -j	m, -o, n, -l
M''	m, -i, n, -j	m, -i, -k, -l	m, -i, -k, -j

M	m j n o	
M'	m, -o, -k, -l	m, -o, l, -j
M''	m, -i, l, -j	m, -i, -k, -l

$PL(7,2)$ and the Golomb-Welch Conjecture

Thank You