Approximating and decomposing clutters with matroids

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The question

Let Λ be a clutter

(aka antichain: a family of mutually incomparable subsets of Ω)

How does Λ ressemble a matroid? *or* How far is Λ from being a matroid? *or* Which is the matroid closest to Λ?

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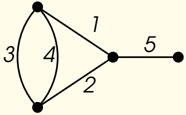
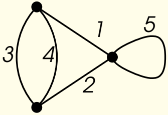
Example: let $\Lambda = \{1\,2\,3, 1\,2\,4, 3\,4\,5\}$

- Both $\{1\,2\,3, 1\,2\,4, 3\,4\}$ and $\{1\,2\,3, 1\,2\,4, 3\,4, 5\}$ are are clutters of circuits of a matroid:





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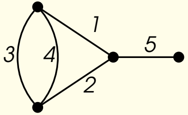
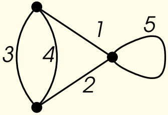
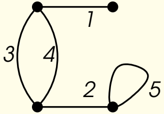
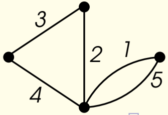




- Both $\{1\,2\,3, 1\,2\,4\}$ and $\{1\,2\,3, 1\,2\,4, 3\,4\,5, 1\,3\,4, 2\,3\,5, 2\,4\,5\}$ are clutters of bases of a matroid:





What's ahead

- State of the art
- Some definitions and notations
- Our results (joint with Jaume Martí-Farré from UPC)
  - Existence theorems
  - Constructive algorithms
- Perspective

Previous efforts/interest in linking clutters and matroids

- Vaderlind, 1986: *Clutters and semimatroids*
- Dress and Wenzel, 1990: *Matroidizing set systems: a new approach to matroid theory*
- Cordovil, Fukuda, and Moreira, 1991: *Clutters and matroids*
- Traldi, 1997-2003:*Clutters and circuits I, II, III*
- Blasiak, Rowe, Traldi, and Yacobi, 2005: *Several definitions of matroids*
- Ford, 2012:*Question on* `mathoverflow.net`: I have a bunch of k-element subsets of $\{1, \ldots, n\}$. Call a matroid good if all of these k-element sets are not bases. I want to find all the matroids M which are minimal among the good ones, in the sense that there is no good matroid whose independent sets are a proper subset of those of M.

Definitions: matroids

A matroid is a pair $(\Omega, \mathcal{C})$ where Ω is a finite set and $\mathcal{C}$ is a family of subsets of Ω, called circuits, satisfying

- $\mathcal{C}$ is a clutter different from $\{\emptyset\}$
- for distinct $C_1, C_2 \in \mathcal{C}$ and $e \in C_1 \cap C_2$, there is $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$

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Subsets of Ω that do not contain any circuit are called independent and subsets that do contain some circuit are dependent

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Moreover,

Fact: a family $\mathcal{B} \subseteq 2^{\Omega}$ is the set of bases of some matroid if, and only if,

- $\mathcal{B} \neq \emptyset$
- for $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, there is $y \in B_2 - B_1$ such that $(B_1 - x) \cup y \in \mathcal{B}$

Definitions: clutters

Denote by $\mathrm{Clutt}(\Omega)$ the set of all clutters on Ω

For $\Lambda \in \mathrm{Clutt}(\Omega)$, let

$$\Lambda^+ = \{B \subseteq \Omega : B \supseteq A \text{ for some } A \in \Lambda\}$$
$$\Lambda^- = \{B \subseteq \Omega : B \subseteq A \text{ for some } A \in \Lambda\}$$

Hence

$$\Lambda = \mathrm{minimal}(\Lambda^+) = \mathrm{maximal}(\Lambda^-)$$

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Define the following two partial orders on $\mathrm{Clutt}(\Omega)$

$$\begin{aligned} \Lambda_1 \leqslant^+ \Lambda_2 &\iff \Lambda_1^+ \subseteq \Lambda_2^+ \\ &\iff \forall\ A \in \Lambda_1\ \exists\ A' \in \Lambda_2\ s.t.\ A \supseteq A' \\ \Lambda_1 \leqslant^- \Lambda_2 &\iff \Lambda_1^- \subseteq \Lambda_2^- \\ &\iff \forall\ A \in \Lambda_1\ \exists\ A' \in \Lambda_2\ s.t.\ A \subseteq A' \end{aligned}$$

Definitions: clutters

Ex: for matroids M_1, M_2

$$\begin{aligned}
\mathcal{C}(M_1) \leqslant^+ \mathcal{C}(M_2) &\Leftrightarrow \text{ every circuit of } M_1 \text{ contains a circuit of } M_2 \\
&\Leftrightarrow (M_1 \text{ is above } M_2 \text{ in the weak order}) \\
\mathcal{B}(M_1) \leqslant^- \mathcal{B}(M_2) &\Leftrightarrow \text{ every basis of } M_1 \text{ is contained in a basis of } M_2 \\
&\Leftrightarrow (M_1 \text{ is below } M_2 \text{ in the weak order})
\end{aligned}$$

(Note: $\mathcal{B}(M_1) \leqslant^+ \mathcal{B}(M_2)$ can also be interpreted in terms of the weak order, but $\mathcal{C}(M_1) \leqslant^- \mathcal{C}(M_2)$ cannot)

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Facts:
$\mathcal{B}(M)^c$ is the clutter of bases of a matroid M^*, called the dual matroid of M
$b(\mathcal{C}(M)) = \mathcal{B}(M^*)$

Answering our initial question

Given $\Lambda \in \mathrm{Clutt}(\Omega)$:

- We want it close to a matroid clutter. Do we choose circuit clutters or basis clutters? Let's say we choose $\Sigma \subseteq \mathrm{Clutt}(\Omega)$

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- And we want our clutter Λ to be above or below with respect to $\textsc{order}$? Let's say we take $\textsc{side}$

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- We want it close to a matroid clutter. Do we choose circuit clutters or basis clutters? Let's say we choose $\Sigma \subseteq \text{Clutt}(\Omega)$
- Now, which order do we use to compare? $\leqslant^+$ or $\leqslant^-$? Let's say we take ORDER
- And we want our clutter Λ to be above or below with respect to ORDER? Let's say we take SIDE

Thm (Informal)
For any choice of ORDER and SIDE, there is a family of clutters $\mathcal{F} \subset \text{Clutt}(\Omega)$ such that:
If $\mathcal{F} \subseteq \Sigma$, then there exist $\Lambda_1, \ldots, \Lambda_s$ in Σ that are closest to Λ with respect to ORDER and SIDE. Moreover, Λ can be recovered from $\Lambda_1, \ldots, \Lambda_s$

Decomposition theorems: in general

Thm Let $\Lambda \in \mathrm{Clutt}(\Omega)$ and $\Sigma \subseteq \mathrm{Clutt}(\Omega)$
If for all $S = \{x_1, \ldots, x_r\} \subseteq \Omega$ the clutter $\Lambda_S = \{\{x_1\}, \ldots, \{x_r\}\}$ belongs to Σ, then

(1) there exists some $\Lambda' \in \Sigma$ such that $\Lambda \leqslant^{+} \Lambda'$

(2) if $\Lambda_1, \ldots, \Lambda_s \in \Sigma$ are the minimal clutters in (1) then $\Lambda = \mathrm{minimal}\{A_1 \cup \cdots \cup A_s : A_i \in \Lambda_i\}$

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By minimality, there is some k such that $\Lambda_k \leqslant^+ \widetilde{\Lambda}$
As $A_0 = A_1 \cup \cdots \cup A_s$ with $A_i \in \Lambda_i$, the set $A_k \in \Lambda_k$ must contain $\{x\}$ for some $x \notin A_0$, a contradiction

Decomposition theorems: in general

For each choice of ORDER and SIDE, we have a "special family of clutters" $\mathcal{F}$ and a decomposition formula

Thm

$\boxed{\leqslant^+,\ \text{above}}$

$$\mathcal{F} = \{\{\{x_1\}, \ldots, \{x_r\}\} : x_1, \ldots, x_r \in \Omega\}$$
$$\Lambda = \text{minimal}\{A_1 \cup \cdots \cup A_s : A_i \in \Lambda_i\}$$

$\boxed{\leqslant^-,\ \text{above}}$

$$\mathcal{F} = \{\{\Omega \setminus x_1, \ldots, \Omega \setminus x_r\} : x_1, \ldots, x_r \in \Omega\}$$
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$\boxed{\leqslant^+,\ \text{below}}$

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Decomposition theorems: matroids

Recall the "special families":

$\mathcal{F}_1 = \{\{\{x_1\}, \ldots, \{x_r\}\} : x_1, \ldots, x_r \in \Omega\}$

$\mathcal{F}_2 = \{\{\Omega \setminus x_1, \ldots, \Omega \setminus x_r\} : x_1, \ldots, x_r \in \Omega\}$

$\mathcal{F}_3 = \{\{x_1 \ldots x_r\} : x_1, \ldots, x_r \in \Omega\}$

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Lem All clutters in $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are clutters of bases

$\mathcal{F}_1$: r elements in parallel
$\mathcal{F}_2$: a cycle on r elements
$\mathcal{F}_3$: a tree on r elements

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Lem All clutters in $\mathcal{F}_1, \mathcal{F}_3$ are clutters of circuits, but $\mathcal{F}_2$ is not

$\mathcal{F}_1$: r loops

$\mathcal{F}_3$: a cycle on r elements

Decomposition theorems: matroids

Obs: $\{\Omega \setminus x_1, \ldots, \Omega \setminus x_r\}$ is not a clutter of circuits if $r \geq 2$, as:
for $e \in (\Omega \setminus x_1) \cap (\Omega \setminus x_2)$, we have

$$(\Omega \setminus x_1 \cup \Omega \setminus x_2) \setminus e = \Omega \setminus e$$

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Ex: $\Lambda = \{1\,2, 1\,3\}$

We look for M such that $\Lambda \leqslant^- \mathcal{C}(M)$, that is, every subset of Λ is contained in a circuit of M

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Two options: $\mathcal{C}(M_1) = \{1\,2\,3\}$ and $\mathcal{C}(M_2) = \{1\,2, 1\,3, 2\,3\}$
As $\mathcal{C}(M_2) \leqslant^- \mathcal{C}(M_1)$, there is only one clutter of circuits above Λ in the order $\leqslant^-$
But $\Lambda \neq \mathcal{C}(M_2)$, so there is no decomposition result

Decomposition theorems: matroids

In summary, given Λ we can consider approximations for any combination of

$$\textsc{circuits} \ / \ \textsc{bases}; \quad \leqslant^{+} \ / \leqslant^{-}; \quad \textsc{above} \ / \ \textsc{below}$$

except: $\textsc{circuits}$, $\leqslant^{-}$, $\textsc{above}$

Decomposition theorems: matroids

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For each choice, the resulting closest clutters $\Lambda_1, \ldots, \Lambda_s$ are called the completions of Λ

Decomposition theorems: matroids

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For each choice, the resulting closest clutters $\Lambda_1, \ldots, \Lambda_s$ are called the completions of Λ

Now, how can we effectively compute these completions?

In principle we should consider 7 cases, but we will see that they actually reduce to only 3

Finding the completions: reductions

Recall:

Lem $\Lambda_1 \leqslant^+ \Lambda_2 \Leftrightarrow \Lambda_1^c \leqslant^- \Lambda_2^c$
$\Lambda_1 \leqslant^+ \Lambda_2 \Leftrightarrow b(\Lambda_2) \leqslant^+ b(\Lambda_1)$

Facts: $\mathcal{B}(M)^c = \mathcal{B}(M^*)$, $b(\mathcal{C}(M)) = \mathcal{B}(M^*)$

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Facts: $\mathcal{B}(M)^c = \mathcal{B}(M^*)$, $b(\mathcal{C}(M)) = \mathcal{B}(M^*)$

By combining blockers and complements, it is enough to solve one of
(CIRCUITS, $\leqslant^+$, ABOVE), (BASES, $\leqslant^+$, BELOW), (BASES, $\leqslant^-$, BELOW)

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Finding the completions: algorithms

We describe the case $(\textsc{circuits}, \leqslant^+, \textsc{above})$

from Martí-Farré 2014

Finding the completions: algorithms

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For distinct $A_1, A_2 \in \Lambda$, define

$$I_\Lambda(A_1 \cup A_2) = \bigcap_{X \in \Lambda, X \subseteq A_1 \cup A_2} X$$

Fact: Λ is the clutter of circuits of some matroid if, and only if, $I_\Lambda(A_1 \cup A_2) = \emptyset$ for all $A_1 \neq A_2 \in \Lambda$

Finding the completions: algorithms

We define three transformations of a clutter Λ

(α) take $A_1 \neq A_2$ with $I_\Lambda(A_1 \cup A_2) \neq \emptyset$ and set

$$\Lambda^\alpha = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\})$$

(β) $\Lambda^\beta = \text{minimal}(\Lambda \cup \{(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2) : A_1 \neq A_2 \in \Lambda\})$

(γ) $\Lambda^\gamma = \text{minimal}(\Lambda \cup \{(A_1 \cup A_2) \setminus x : x \in A_1 \cap A_2, A_1 \neq A_2 \in \Lambda\})$

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Lem $\Lambda \leqslant^+ \Lambda^\alpha,\ \Lambda \leqslant^+ \Lambda^\beta,\ \Lambda \leqslant^+ \Lambda^\gamma$

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Lem $\Lambda \leqslant^+ \Lambda^\alpha,\ \Lambda \leqslant^+ \Lambda^\beta,\ \Lambda \leqslant^+ \Lambda^\gamma$

Lem None of the transformations applied to Λ yields a different clutter if and only if Λ is the clutter of circuits of some matroid

Finding the completions: algorithms

Thm Let Λ' be a completion of Λ for ($\textsc{circuits}$, $\leqslant^+$, $\textsc{above}$). There is a finite sequence $(i_1, \ldots, i_k) \in \{\alpha, \beta, \gamma\}^k$ such that

$$\Lambda' = \Lambda^{i_1, \ldots, i_k}$$

Finding the completions: algorithms

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So to obtain all completions, we apply the α, β, γ transformations in all possible orders until no more applications are possible, and take the minimal clutters thus generated

Finding the completions: algorithms

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So to obtain all completions, we apply the α, β, γ transformations in all possible orders until no more applications are possible, and take the minimal clutters thus generated

We have similar looking algorithms for $(\textsc{circuits}, \leqslant^+, \textsc{below})$ and $(\textsc{circuits}, \leqslant^-, \textsc{below})$

Finding the completions: example

$\Lambda^{\alpha} = \mathsf{minimal}(\Lambda \cup \{A_1 \cap A_2\})$ if $I_{\Lambda}(A_1 \cup A_2) \neq \emptyset$
$\Lambda^{\beta} = \mathsf{minimal}(\Lambda \cup \{(A_1 \cup A_2) \setminus I_{\Lambda}(A_1 \cup A_2) : A_1 \neq A_2 \in \Lambda\})$
$\Lambda^{\gamma} = \mathsf{minimal}(\Lambda \cup \{(A_1 \cup A_2) \setminus x : x \in A_1 \cap A_2, A_1 \neq A_2 \in \Lambda\})$

$\Lambda = \{123, 124, 345\}$:

Finding the completions: example

$$\Lambda^\alpha = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\}) \text{ if } I_\Lambda(A_1 \cup A_2) \neq \emptyset$$
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$\Lambda = \{123, 124, 345\}$: $I_\Lambda(123 \cup 124) = \{12\}$

Finding the completions: example

$\Lambda^\alpha = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\})$ if $I_\Lambda(A_1 \cup A_2) \neq \emptyset$
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$\Lambda = \{123, 124, 345\}$: $I_\Lambda(123 \cup 124) = \{12\}$

$\Lambda^\alpha = \text{minimal}(123, 124, 345, 12) = \{12, 345\} = \mathcal{C}(M_1)$

Finding the completions: example

$$\Lambda^\alpha = \mathrm{minimal}(\Lambda \cup \{A_1 \cap A_2\}) \text{ if } I_\Lambda(A_1 \cup A_2) \neq \emptyset$$
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Finding the completions: example

$$\Lambda^\alpha = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\}) \text{ if } I_\Lambda(A_1 \cup A_2) \neq \emptyset$$
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$$\Lambda^\gamma = \text{minimal}(123, 124, 345, 234, 134, 1245, 1235)$$
$$= \{123, 124, 345, 234, 134\}$$

Finding the completions: example

$\Lambda^\alpha = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\})$ if $I_\Lambda(A_1 \cup A_2) \neq \emptyset$
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$\Lambda = \{123, 124, 345\}$: $I_\Lambda(123 \cup 124) = \{12\}$

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$$\Lambda^\gamma = \text{minimal}(123, 124, 345, 234, 134, 1245, 1235) = \{123, 124, 345, 234, 134\}$$

$I_{\Lambda^\gamma}(123 \cup 124) = \{12\}$, $I_{\Lambda^\gamma}(345 \cup 234) = \{34\}$,
$I_{\Lambda^\gamma}(345 \cup 134) = \{34\}$

Finding the completions: example

$$\Lambda^\alpha = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\}) \text{ if } I_\Lambda(A_1 \cup A_2) \neq \emptyset$$
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$\Lambda = \{123, 124, 345\}$: $I_\Lambda(123 \cup 124) = \{12\}$

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$$I_{\Lambda^\gamma}(123 \cup 124) = \{12\},\ I_{\Lambda^\gamma}(345 \cup 234) = \{34\},$$
$$I_{\Lambda^\gamma}(345 \cup 134) = \{34\}$$

$\Lambda^{\gamma,\alpha_1} = \text{minimal}(123, 124, 345, 234, 134, 12) = \{12, 345, 234, 134\}$

Finding the completions: example

$$\Lambda^\alpha = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\}) \text{ if } I_\Lambda(A_1 \cup A_2) \neq \emptyset$$
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$\Lambda^\gamma = \text{minimal}(123, 124, 345, 234, 134, 1245, 1235)$
$= \{123, 124, 345, 234, 134\}$

$I_{\Lambda^\gamma}(123 \cup 124) = \{12\}$, $I_{\Lambda^\gamma}(345 \cup 234) = \{34\}$,
$I_{\Lambda^\gamma}(345 \cup 134) = \{34\}$

$\Lambda^{\gamma,\alpha_1} = \text{minimal}(123, 124, 345, 234, 134, 12) = \{12, 345, 234, 134\}$
As $\Lambda^\alpha \leqslant^+ \Lambda^{\gamma,\alpha_1}$, there is no need to continue from here

Finding the completions: example

$\Lambda^\alpha = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\})$ if $I_\Lambda(A_1 \cup A_2) \neq \emptyset$
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$\Lambda^\gamma = \text{minimal}(\Lambda \cup \{(A_1 \cup A_2) \setminus x : x \in A_1 \cap A_2, A_1 \neq A_2 \in \Lambda\})$

$\Lambda^\gamma = \{123, 124, 345, 234, 134\}$

$I_{\Lambda^\gamma}(123 \cup 124) = \{12\}$, $I_{\Lambda^\gamma}(345 \cup 234) = \{34\}$,
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Finding the completions: example

$\Lambda^{\alpha} = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\})$ if $I_{\Lambda}(A_1 \cup A_2) \neq \emptyset$
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$\Lambda^{\gamma} = \text{minimal}(\Lambda \cup \{(A_1 \cup A_2) \setminus x : x \in A_1 \cap A_2, A_1 \neq A_2 \in \Lambda\})$

$\Lambda^{\gamma}=\{123, 124, 345, 234, 134\}$

$I_{\Lambda^{\gamma}}(123 \cup 124) = \{12\}$, $I_{\Lambda^{\gamma}}(345 \cup 234) = \{34\}$,
$I_{\Lambda^{\gamma}}(345 \cup 134) = \{34\}$

$\Lambda^{\gamma,\alpha_2}=\text{minimal}(123, 124, 345, 234, 134, 34) = \{123, 124, 34\} = \Lambda^{\beta}$

Finding the completions: example

$\Lambda^\alpha = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\})$ if $I_\Lambda(A_1 \cup A_2) \neq \emptyset$
$\Lambda^\beta = \text{minimal}(\Lambda \cup \{(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2) : A_1 \neq A_2 \in \Lambda\})$
$\Lambda^\gamma = \text{minimal}(\Lambda \cup \{(A_1 \cup A_2) \setminus x : x \in A_1 \cap A_2, A_1 \neq A_2 \in \Lambda\})$

$\Lambda^\gamma = \{123, 124, 345, 234, 134\}$
$I_{\Lambda^\gamma}(123 \cup 124) = \{12\}$, $I_{\Lambda^\gamma}(345 \cup 234) = \{34\}$,
$I_{\Lambda^\gamma}(345 \cup 134) = \{34\}$

$\Lambda^{\gamma,\alpha_2} = \text{minimal}(123, 124, 345, 234, 134, 34) = \{123, 124, 34\} = \Lambda^\beta$
$\Lambda^{\gamma,\alpha_3} = \text{minimal}(123, 124, 345, 234, 134, 34) = \{123, 124, 34\} = \Lambda^\beta$

Finding the completions: example

$$\Lambda^\alpha = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\}) \text{ if } I_\Lambda(A_1 \cup A_2) \neq \emptyset$$
$$\Lambda^\beta = \text{minimal}(\Lambda \cup \{(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2) : A_1 \neq A_2 \in \Lambda\})$$
$$\Lambda^\gamma = \text{minimal}(\Lambda \cup \{(A_1 \cup A_2) \setminus x : x \in A_1 \cap A_2, A_1 \neq A_2 \in \Lambda\})$$

$\Lambda^\gamma = \{123, 124, 345, 234, 134\}$

$I_{\Lambda^\gamma}(123 \cup 124) = \{12\}$, $I_{\Lambda^\gamma}(345 \cup 234) = \{34\}$,
$I_{\Lambda^\gamma}(345 \cup 134) = \{34\}$

$\Lambda^{\gamma,\alpha_2} = \text{minimal}(123, 124, 345, 234, 134, 34) = \{123, 124, 34\} = \Lambda^\beta$
$\Lambda^{\gamma,\alpha_3} = \text{minimal}(123, 124, 345, 234, 134, 34) = \{123, 124, 34\} = \Lambda^\beta$

$\Lambda^{\gamma,\beta} = \text{minimal}(123, 124, 345, 234, 134, 34, 25, 15)$
$= \{123, 124, 34, 15, 25\}$

Finding the completions: example

$$\Lambda^{\alpha} = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\}) \text{ if } I_{\Lambda}(A_1 \cup A_2) \neq \emptyset$$
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$\Lambda^{\gamma} = \{123, 124, 345, 234, 134\}$

$I_{\Lambda^{\gamma}}(123 \cup 124) = \{12\}$, $I_{\Lambda^{\gamma}}(345 \cup 234) = \{34\}$,
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$\Lambda^{\gamma,\alpha_2} = \text{minimal}(123, 124, 345, 234, 134, 34) = \{123, 124, 34\} = \Lambda^{\beta}$
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$\Lambda^{\gamma,\beta} = \text{minimal}(123, 124, 345, 234, 134, 34, 25, 15)$
$= \{123, 124, 34, 15, 25\}$

Like before, $\Lambda^{\beta} \leqslant^{+} \Lambda^{\gamma,\beta}$, so no need to continue this path

Finding the completions: example

$\Lambda^\alpha = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\})$ if $I_\Lambda(A_1 \cup A_2) \neq \emptyset$
$\Lambda^\beta = \text{minimal}(\Lambda \cup \{(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2) : A_1 \neq A_2 \in \Lambda\})$
$\Lambda^\gamma = \text{minimal}(\Lambda \cup \{(A_1 \cup A_2) \setminus x : x \in A_1 \cap A_2, A_1 \neq A_2 \in \Lambda\})$

$\Lambda^\gamma = \{123, 124, 345, 234, 134\}$
$I_{\Lambda^\gamma}(123 \cup 124) = \{12\}$, $I_{\Lambda^\gamma}(345 \cup 234) = \{34\}$,
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$\Lambda^{\gamma,\alpha_2} = \text{minimal}(123, 124, 345, 234, 134, 34) = \{123, 124, 34\} = \Lambda^\beta$
$\Lambda^{\gamma,\alpha_3} = \text{minimal}(123, 124, 345, 234, 134, 34) = \{123, 124, 34\} = \Lambda^\beta$

$\Lambda^{\gamma,\beta} = \text{minimal}(123, 124, 345, 234, 134, 34, 25, 15)$
$= \{123, 124, 34, 15, 25\}$
Like before, $\Lambda^\beta \leqslant^+ \Lambda^{\gamma,\beta}$, so no need to continue this path

$\Lambda^{\gamma,\gamma} = \{123, 124, 345, 234, 134, 235, 245, 135, 145\}$

Finding the completions: example

$\Lambda^\alpha = \text{minimal}(\Lambda \cup \{A_1 \cap A_2\})$ if $I_\Lambda(A_1 \cup A_2) \neq \emptyset$
$\Lambda^\beta = \text{minimal}(\Lambda \cup \{(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2) : A_1 \neq A_2 \in \Lambda\})$
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$\Lambda^\gamma = \{123, 124, 345, 234, 134\}$
$I_{\Lambda^\gamma}(123 \cup 124) = \{12\}$, $I_{\Lambda^\gamma}(345 \cup 234) = \{34\}$,
$I_{\Lambda^\gamma}(345 \cup 134) = \{34\}$

$\Lambda^{\gamma,\alpha_2} = \text{minimal}(123, 124, 345, 234, 134, 34) = \{123, 124, 34\} = \Lambda^\beta$
$\Lambda^{\gamma,\alpha_3} = \text{minimal}(123, 124, 345, 234, 134, 34) = \{123, 124, 34\} = \Lambda^\beta$

$\Lambda^{\gamma,\beta} = \text{minimal}(123, 124, 345, 234, 134, 34, 25, 15)$
$= \{123, 124, 34, 15, 25\}$
Like before, $\Lambda^\beta \leqslant^+ \Lambda^{\gamma,\beta}$, so no need to continue this path

$\Lambda^{\gamma,\gamma} = \{123, 124, 345, 234, 134, 235, 245, 135, 145\}$

...

$\Lambda^{\gamma,\gamma,\gamma} = \{123, 124, 345, 234, 134, 235, 245, 135, 145, 125\}$

Finding the completions: example

Hence, the smallest circuit clutters that are larger than $\{123, 124, 345\}$ with respect to the order $\leqslant^+$ are

$$\Lambda_1 = \{12, 345\}$$
$$\Lambda_2 = \{123, 124, 34\}$$
$$\Lambda_3 = \{123, 124, 345, 234, 134, 235, 245, 135, 145, 125\}$$

Finding the completions: example

Hence, the smallest circuit clutters that are larger than $\{123, 124, 345\}$ with respect to the order $\leqslant^+$ are

$$\Lambda_1 = \{12, 345\}$$
$$\Lambda_2 = \{123, 124, 34\}$$
$$\Lambda_3 = \{123, 124, 345, 234, 134, 235, 245, 135, 145, 125\}$$

We have also $\Lambda = \text{minimal}(A_1 \cup A_2 \cup A_3, A_i \in \Lambda_i)$

Finding the completions: example

Hence, the smallest circuit clutters that are larger than $\{123, 124, 345\}$ with respect to the order $\leqslant^+$ are

$$\Lambda_1 = \{12, 345\}$$
$$\Lambda_2 = \{123, 124, 34\}$$
$$\Lambda_3 = \{123, 124, 345, 234, 134, 235, 245, 135, 145, 125\}$$

We have also $\Lambda = \text{minimal}(A_1 \cup A_2 \cup A_3, A_i \in \Lambda_i)$

But actually $\Lambda = \text{minimal}(A_1 \cup A_2, A_i \in \Lambda_i)$

It does happen very often that not all the clutters of the completion are needed for the decomposition to hold

Other families of clutters

Let $\mathcal{N}$ be a class of matroids (as binary, graphic, transversal, ...)

Define:

$$\Sigma(\mathcal{C},\mathcal{N}) = \{\mathcal{C}(M) : M \in \mathcal{N}\}$$
$$\Sigma(\mathcal{B},\mathcal{N}) = \{\mathcal{B}(M) : M \in \mathcal{N}\}$$

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For each ORDER and SIDE, if $\Sigma(\mathcal{C},\mathcal{N})$ or $\Sigma(\mathcal{B},\mathcal{N})$ contain all clutters in the corresponding "special family", then we can approximate any clutter Λ with matroids in the class $\mathcal{N}$

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Fact: for all "usual" classes of matroids, all combinations work except (CIRCUITS, $\leqslant^{-}$, ABOVE)

Other families of clutters: examples

With respect to (CIRCUITS, $\leqslant^+$, ABOVE)

- The matroid $U_{2,4}$ with circuits $\{123, 124, 134, 234\}$ has 6 binary completions:

$$\{i_1 i_2, i_1 i_3 i_4, i_2 i_3 i_4\} \qquad \text{for } \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$$

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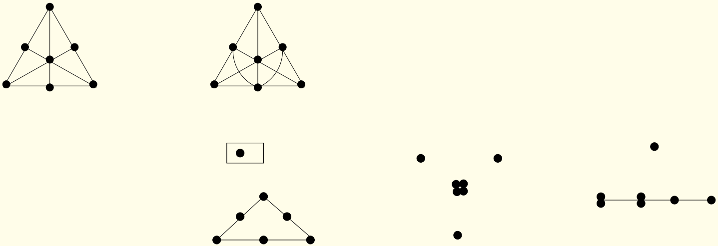
- The non-Fano matroid has 9 completions in the class of matroids representable over a field of characteristic 2



To be done

- Improve the algorithms to compute completions
- Find algorithms to compute completions directly with respect to BASES
- Find algorithms to compute completions in subclasses of matroids
- Study the relationship between the number of completions and the number needed for the decomposition formula to work
- Study the behaviour of deletion and contraction
- Answer the question on `mathoverflow.net`
- Apply the results!