Integer points and Ehrhart polynomial of lattice path matroid polytope

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Ehrhart theory

A Lattice polytope $P \subset \mathbb{R}^d$ is a convex hull of a finite set of points in $\mathbb{Z}^d$. For $k \in \mathbb{Z}_{>0}$ let $L_P(k) := \#(kP \cap \mathbb{Z}^d)$

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Example

$Q_2 = conv\{(0,0),(1,0),(0,1),(1,1)\} = \{x, y \in \mathbb{R} : 0 \leq x, y \leq 1\}$.

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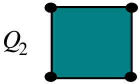

k	1
$L_{Q_2}(k)$	4

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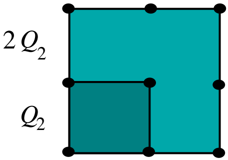

k	1	2
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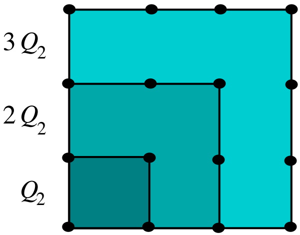

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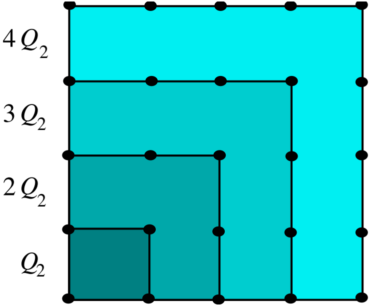

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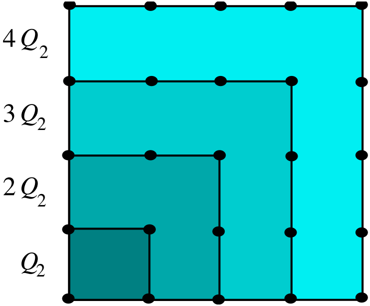

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$$L_{Q_2}(k) = (k+1)^2$$

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d-dimensional cube : $L_{Q_d}(k) = (k+1)^d = \sum_{i=0}^{d} \binom{d}{i} k^i$

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Therefore, $(-1)^{dim(P)} L_P(-k)$ enumerates the interior lattice points in kP.

Permutahedron

The d-dimensional permutahedron P_d is defined as
$P_d := conv\{(\pi(1)-1, \pi(2)-1, \ldots, \pi(d)-1) : \pi \in \mathcal{S}_d\}$

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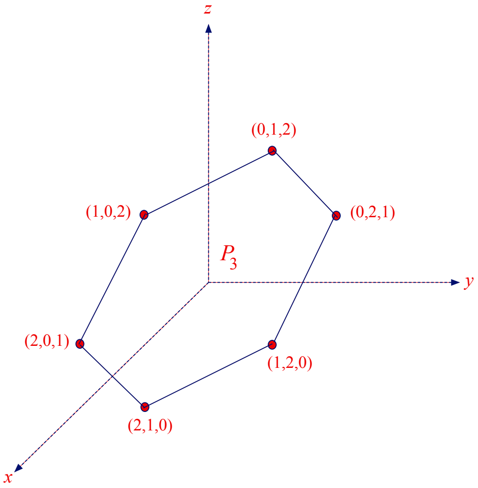

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$$f_d = d^{d-2} = vol(P_d)$$

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$$L_{C_d}(k) = \sum_{i=0}^{d} f_i k^i$$

where $f_i = vol(C_i(t_1, \ldots, t_n))$.

Ehrhart series

The standard d-simplex

$$\begin{aligned}\Delta \quad &= \{\mathbf{x} \in \mathbb{R}^d_{\geq 0} : x_1 + \cdots + x_d \leq 1\} \\ &= conv\{(0, \ldots, 0), (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}\end{aligned}$$

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This motivate to define the Ehrhart serie of the lattice polytope P as

$$Ehr_P(z) := 1 + \sum_{t\geq 1} L_P(t) z^t$$

Ehrhart's theorem (Equivalent) For any lattice polytope P, $Ehr_P(z)$ is a rational function of the form

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- $h_d = \#(P^\circ \cap \mathbb{Z}^d)$ and $h_1 = \#(P \cap \mathbb{Z}^d) - d - 1$
- Theorem (Stanley 1980) $h_0, \ldots, h_d$ are nonnegative integers

Independents

A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set $(E = \{1, \ldots, n\})$ and $\mathcal{I}$ is a family of subsets of E verifying the following conditions :

($I1$) $\emptyset \in \mathcal{I}$,

($I2$) If $I \in \mathcal{I}$ and $I' \subset I$ then $I' \in \mathcal{I}$,

($I3$) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$.

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The rank of a set $X \subseteq E$ is defined by

$$r_M(X) = \max\{|Y| : Y \subseteq X, Y \in \mathcal{I}\}.$$

Bases

A base of a matroid is a maximal independent set.
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We denote by $\mathcal{B}$ the set of all bases of a matroid.

Lemma All the bases of a matroid have the same cardinality r.

The rank of a matroid M, denoted by $r(M)$, is the rank of one of its bases.

The family $\mathcal{B}$ verifies the following conditions :

$(B1)$ $\mathcal{B} \neq \emptyset$,

$(B2)$ (**exchange propety**) $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$ then there exist $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

Theorem $\mathcal{B}$ is the set of basis of a matroid if and only if it verifies $(B1)$ and $(B2)$.

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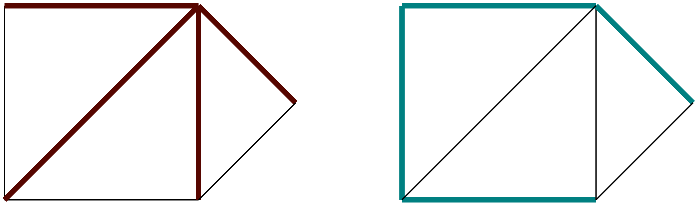
Then, $M(G) = (\mathcal{B}, E)$ is a matroid with $r(M(G)) = n - c$ where c is the number of connected components of G.

Example



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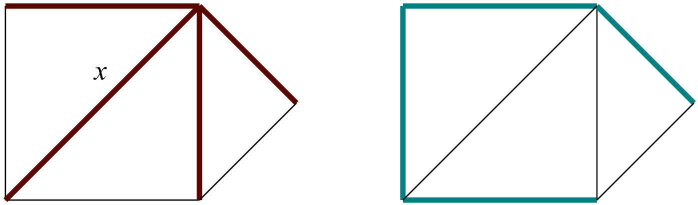

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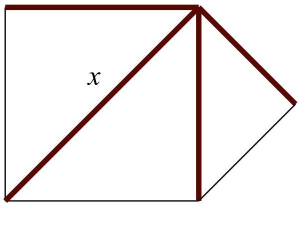

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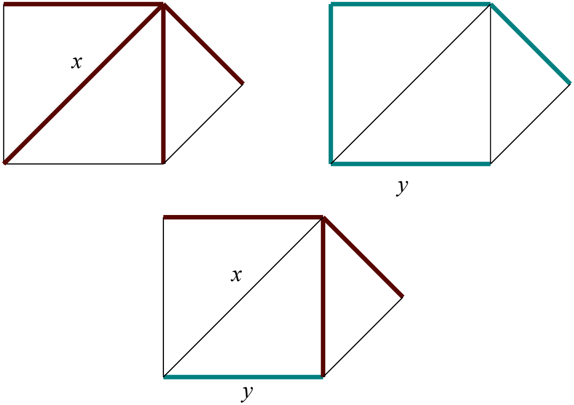

Matroid base polytope

Let $M = (\mathcal{B}, E)$ with $|E| = n$. For each base $B \in \mathcal{B}$, the incident vector $e_B \in \mathbb{R}^E$ is defined by

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where e_i denotes i^{th} standard base vector in $\mathbb{R}^n$.

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Let Δ_E be the simplexe in $\mathbb{R}^E$, i.e.,

$$\Delta_E = conv(e_i : i \in E) = \{x \in \mathbb{R}^E : \sum_{i \in E} x_i = 1,\ x_i \geq 0 \text{ for all } i \in E\}$$

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Theorem For $M = (\mathcal{B}, E)$

- $P_M \subseteq r\Delta_E$ where $r = r(M)$ (implying that $dim(P) \leq n-1$)
- Each edge of P_M is a translation of $conv(e_i, e_j)$ pour $i, j \in E, i \neq j$.

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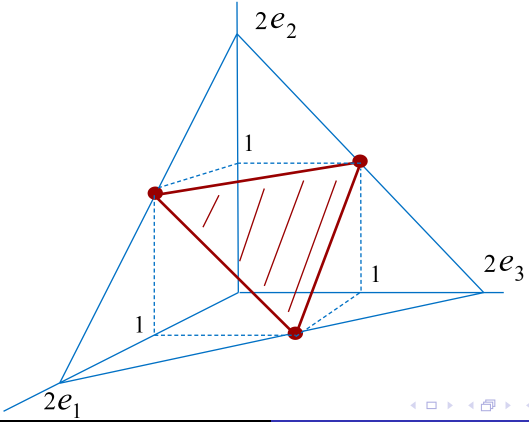
Example $P_{U_{2,3}} = conv\{(1,1,0),(1,0,1),(0,1,1)\}$



Matroid base polytope

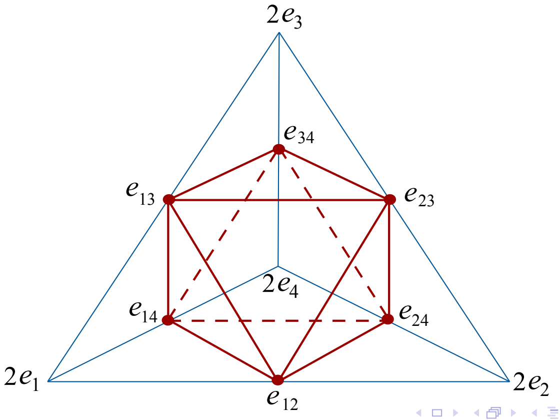
Example

$$P_{U_{2,4}} = conv \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^4$$

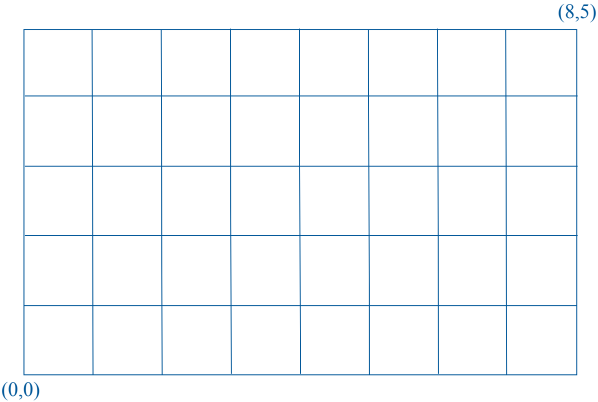
Matroid base polytope

Example

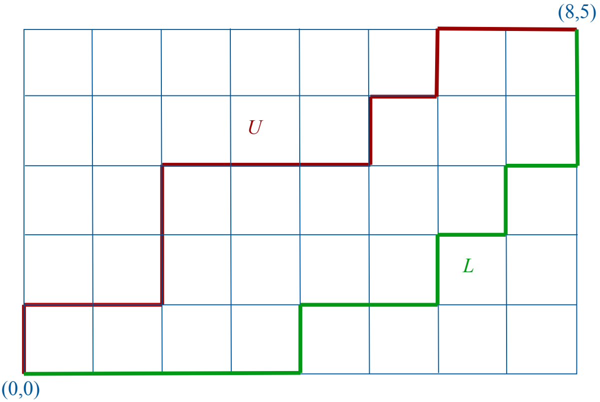
$$P_{U_{2,4}} = conv\left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\} \subset \mathbb{R}^4$$



Lattice path matroid



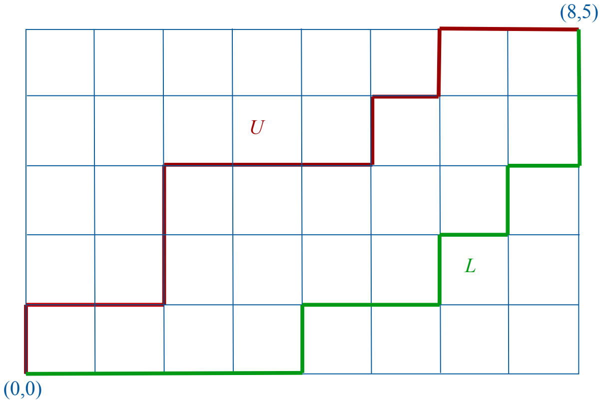
Lattice path matroid



U=NEENNEEENENEE

L=EEEENEENENENN

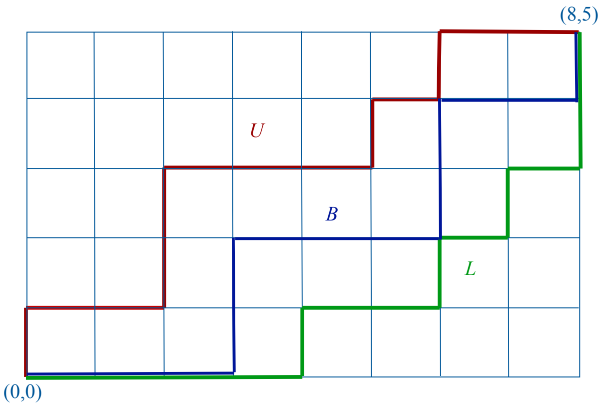
Lattice path matroid



U={1,4,5,9,11}
U=NEENNEEENENEE

L={5,8,10,12,13}
L=EEEENEENENENN

Lattice path matroid

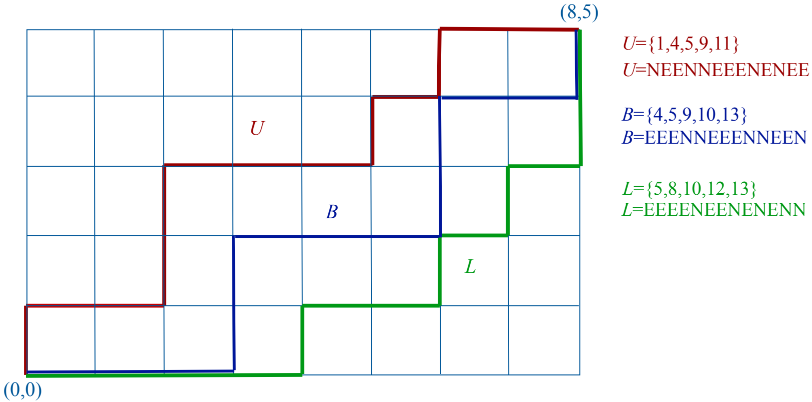


U={1,4,5,9,11}
U=NEENNEEENENEE

B={4,5,9,10,13}
B=EEENNEEENNEEN

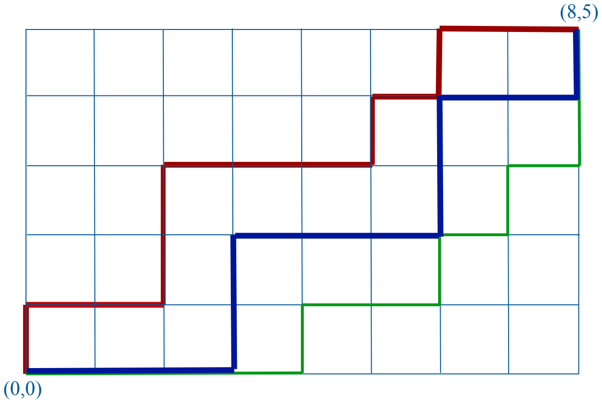
L={5,8,10,12,13}
L=EEEENEENENENN

Lattice path matroid



$M[U, L]$ lattice path matroid (LPM) of rank r (# rows) on $r + m$ (# rows + # columns) elements.

LPM base exchange base



U={1,4,5,9,11}
U=NEENNEEENENEE

B={4,5,9,10,13}
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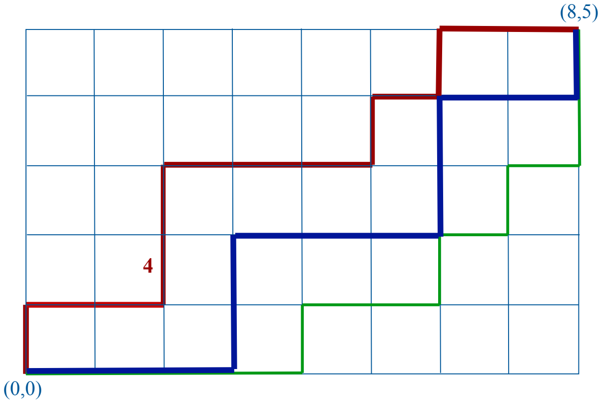
LPM base exchange base



U={1,**4**,5,9,11}
U=NEE**N**NEEENENEE

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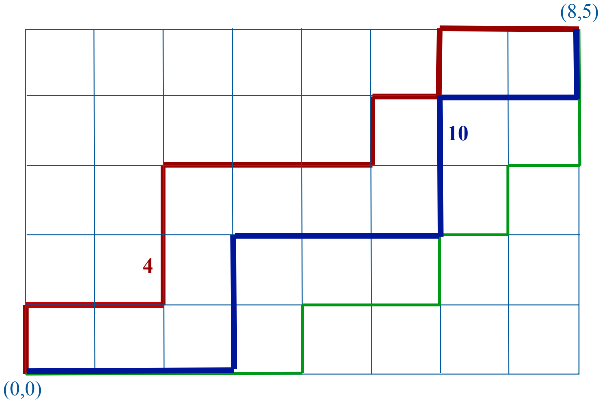
B=EEENNEEENNEEN

LPM base exchange base



U={1,**4**,5,9,11}
U=NEE**N**NEEENENEE

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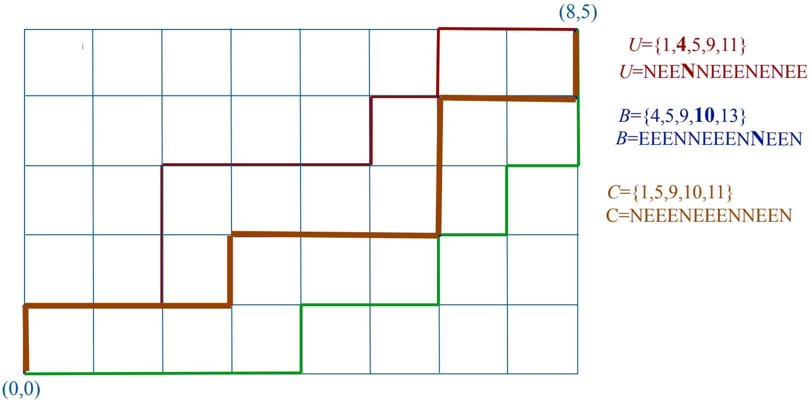
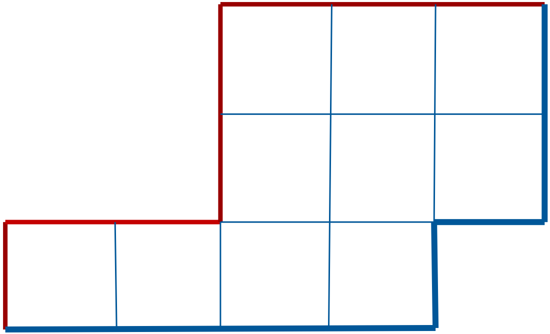
B=EEENNEEEN**N**EEN

LPM base exchange base



Generalized lattice path



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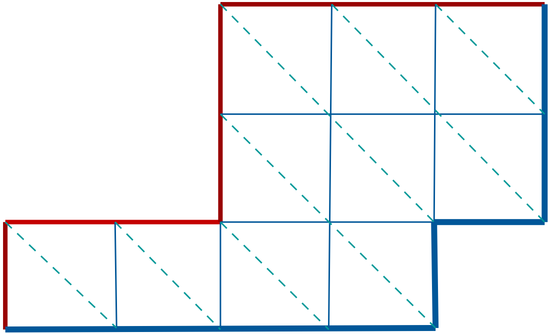

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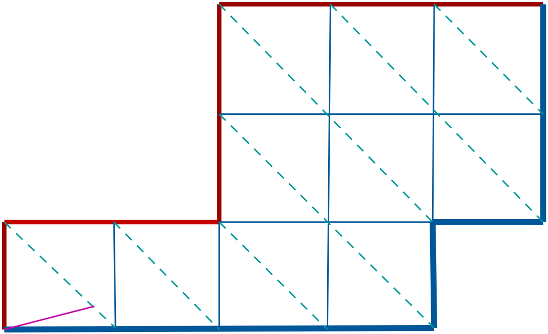

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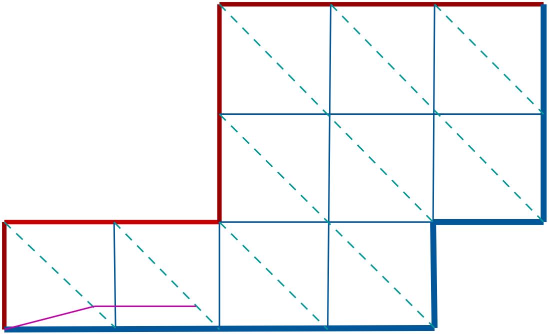
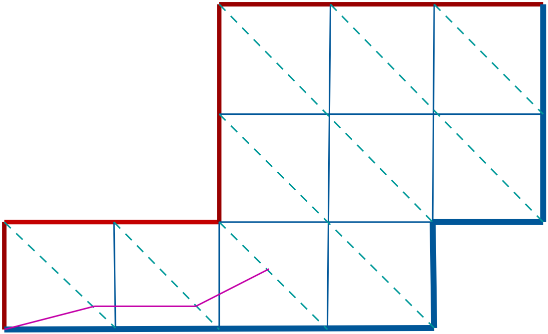
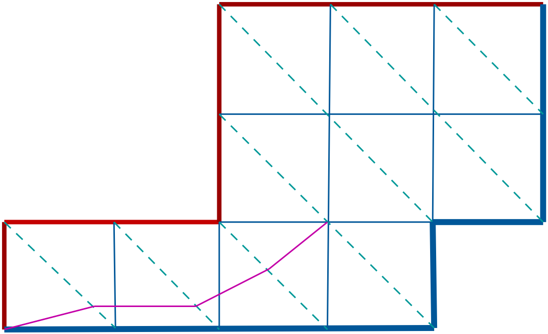


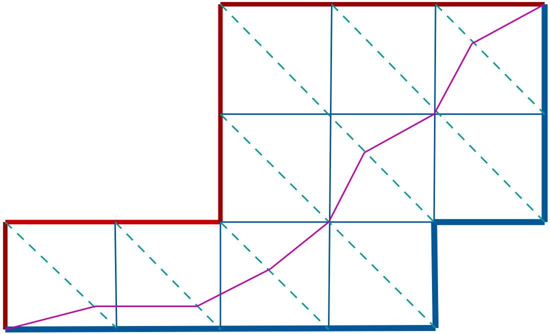
Generalized lattice path



Generalized lattice path

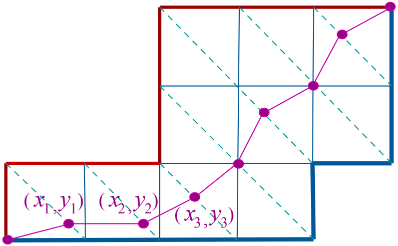


Generalized lattice path



Generalized lattice path

A generalized path P starts at $(0,0)$ and ends at $(r, r+m)$ and it is monotonously increasing $x_i \leq x_{i+1}$ and $y_i \leq y_{i+1}$.



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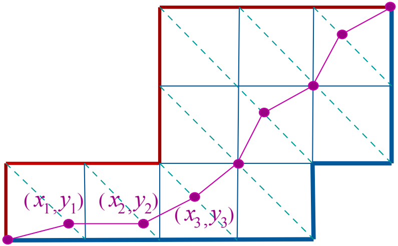

Let $st(P) = (p_1, \dots, p_{r+m})$ where $p_{i+1} = y_{i+1} - y_i$ for each i.

We call $st(P)$ step vector of P.

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M[U, L]$ be a LPM of rank r on $r + m$ elements.
Let $st(L) = (l_1, \ldots, l_{r+m})$ and $st(U) = (u_1, \ldots, u_{r+m})$.

Characterizing step vectors

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$$\mathcal{C}_M = \left\{ p \in \mathbb{R}^{r+m} \mid 0 \leq p_i \leq 1, \ \sum_{j=1}^{i} l_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} u_j \ \forall i \right\}$$

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- $0 \leq y_{i+1} - y_i \leq 1$
- Any generalized path stay between U and L.

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M = M[U, L]$ be a LPM of rank r on $r + m$ elements. Then, $P_M = \mathcal{C}_M$.

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Proof (idea).
$P_M = conv\{$ characteristic vectors of $\mathcal{B}(M)\} \subseteq conv\{\mathcal{C}_M\} = \mathcal{C}_M$.

Points in LPM polytope

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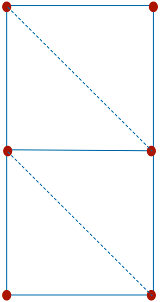
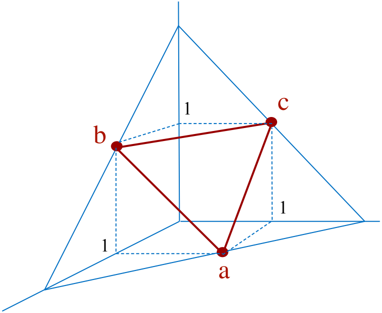
$$kP_M \cap \mathbb{Z}^{r+m} = \mathcal{C}_M^k$$

Integer points in LPM polytopes

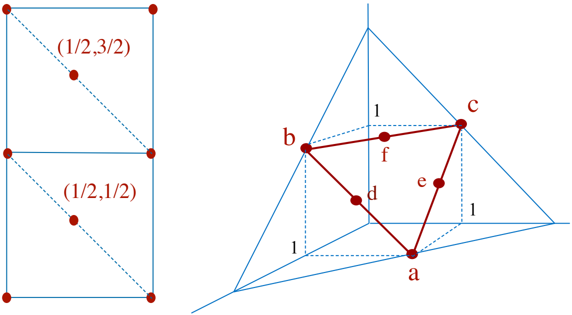
Example : Consider $P_{U_{2,3}}$





a=(1,1,0)
b=(1,0,1)
c=(0,1,1)

Example : Construct paths in $\mathcal{C}^2_{U_{2,3}}$

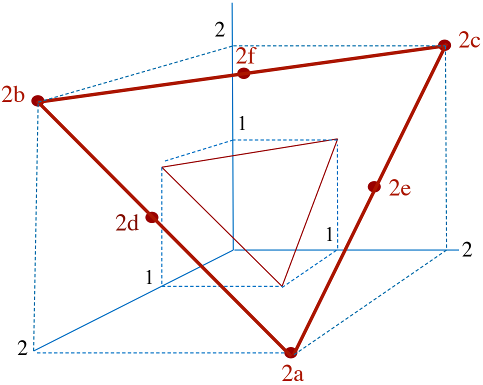


a=(1,1,0) d=(1,1/2,1/2)

b=(1,0,1) e=(1/2,1,1/2)

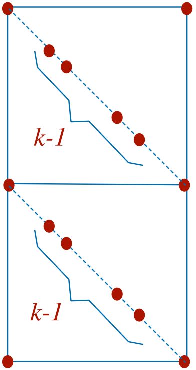
c=(0,1,1) f=(1/2,1/2,1)

Example : $2P_{U_{2,3}}$

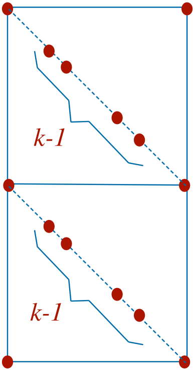


2a=(2,2,0) 2d=(2,1,1)
2b=(2,0,2) 2e=(1,2,1)
2c=(0,2,2) 2f=(1,1,2)

Let us consider $kP_{U_{2,3}}$

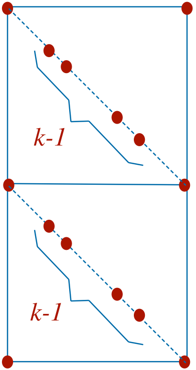


Let us consider $kP_{U_{2,3}}$



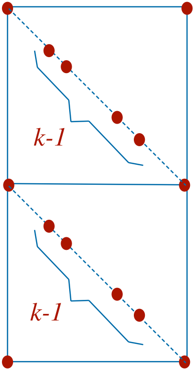
$\mathcal{C}^k_{U_{2,3}}$

Let us consider $kP_{U_{2,3}}$



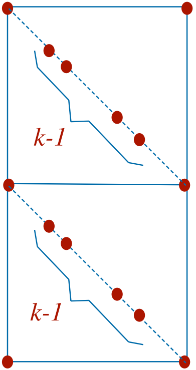
$\mathcal{C}^k_{U_{2,3}} = \frac{1}{2}(k+1)(k+2)$

Let us consider $kP_{U_{2,3}}$



$$\mathcal{C}^k_{U_{2,3}} = \frac{1}{2}(k+1)(k+2) = \frac{1}{2}k^2 + \frac{3}{2}k + 1$$

Let us consider $kP_{U_{2,3}}$



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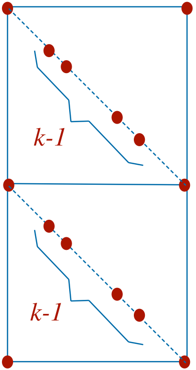
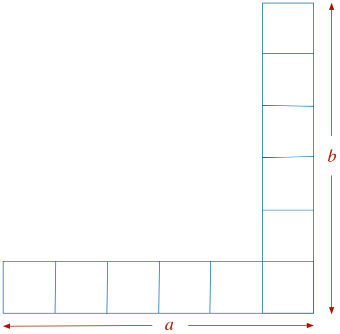


$$\mathcal{C}^k_{U_{2,3}} = \frac{1}{2}(k+1)(k+2) = \frac{1}{2}k^2 + \frac{3}{2}k + 1 = kP_{U_{2,3}} \cap \mathbb{Z}^3 = L_{P_{U_{2,3}}}(k)$$

Let $S(a,b)$ be the matroid associated to



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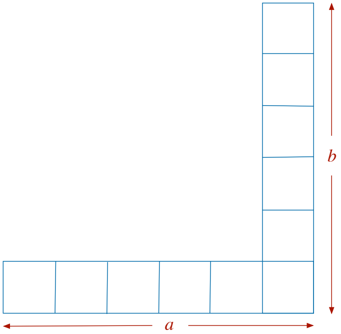

$$L_{S(a,b)}(k) = \sum_{i=0}^{k} \binom{a+k-1-i}{a-1} \binom{b+k-1-i}{b-1}$$

Distributive polytopes

A polytope $P \subseteq \mathbb{R}^n$ is called distributive if for all $x, y \in P$ also their componentwise maximum and minimum $\max(x, y)$ and $\min(x, y)$ are in P.

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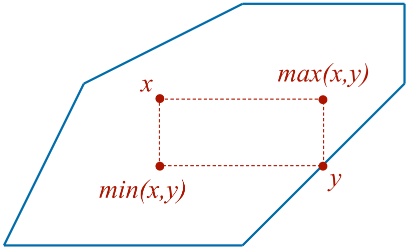
Example : A distributive polytope in $\mathbb{R}^2$.



Distributive polytopes

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M = M[U, L]$ be a rank r LPM on $r + m$ elements (we suppose that M is connected, i.e., $dim(P) = r + m - 1$).

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Let $M = M[U, L]$ be a rank r LPM on $r + m$ elements (we suppose that M is connected, i.e., $dim(P) = r + m - 1$).Then, there exists a bijective affine transformation taking $P_M \subset \mathbb{R}^{r+m}$ into a full-dimensional distributive integer polytope $Q_M \subset \mathbb{R}^{r+m-1}$ such that $L_{P_M}(t) = L_{Q_M}(t)$.

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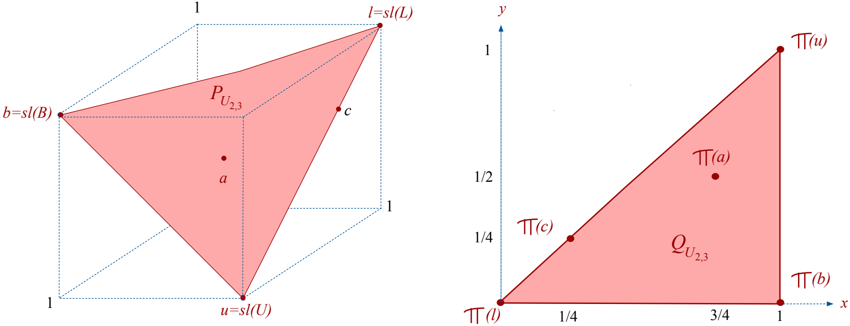
Proof (idea). Check that

$$\begin{array}{llcl} \pi : & P_M \subset \mathbb{R}^{r+m} & \longrightarrow & \mathbb{R}^{r+m-1} \\ & p = (p_1, \ldots, p_{r+m}) & \mapsto & (p_1 - L_1, \ldots, \sum_{j=1}^{r+m-1}(p_j - L_j)) \end{array}$$

is suitable transformation.

Example :



We have $\pi(a) = (\frac{3}{4}, \frac{1}{2}), \pi(b) = (1, 0)$ and $\pi(c) = (\frac{1}{4}, \frac{1}{4})$.

Order polytopes

Let X be a poset on $\{1, \ldots, n\}$ such that this labeling is natural, i.e., if $i <_X j$ then $i < j$.

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The order polytope $\mathcal{O}(X)$ of X is defined as the set of those $x \in \mathbb{R}^n$ such that

$$0 \leq x_i \leq 1, \text{ for all } i \in X \text{ and}$$
$$x_i \geq x_j, \text{ if } i \leq j \text{ in } X$$

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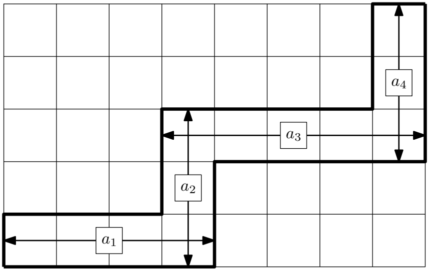
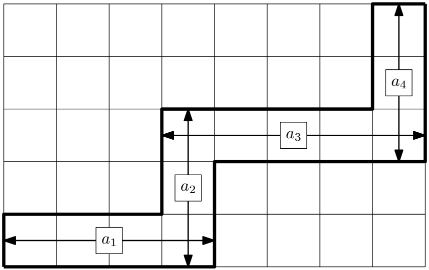
Remark $\mathcal{O}(X)$ is a bounded convex polytope

Snake polytopes



Snake $S(a_1, a_2, a_3, a_4)$



Snake $S(a_1, a_2, a_3, a_4)$

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $a_1, \ldots, a_k \geq 2$ be integers. Then, a connected LPM M is the snake $S(a_1, \ldots a_k)$ if and only if Q_M is the order polytope of the zig-zag chain poset on $a_1, \ldots, a_k$.

Snake polytopes

Recall that the Ehrhart serie is given by

$$Ehr_P(z) = 1 + \sum_{t \geq 1} L_P(t) z^t = \frac{h_s z^s + h_{s-1} z^{s-1} + \cdots + h_0}{(1-z)^{dim(P)+1}}$$

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Conjecture (De Loera, Haws, Köppe, 2009) The h-vector of base matroid polytopes are unimodal, i.e.,

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Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $a, b \geq 2$ be integers. The h-vectors of the snake polytopes $P_{S(a,\ldots,a)}$ and $P_{S(a,b)}$ are unimodal.