Identities in plactic, hypoplactic, sylvester, Baxter, and related monoids

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Young tableaux & Schensted's algorithm

Let $\mathcal{A} = \{1 < 2 < 3 < \ldots\}$ and let $\mathcal{A}_n = \{1 < 2 < 3 < \ldots < n\}$.

1	1	3	4	4
2	3	6		
5				

- Rows weakly increasing left to right.
- Columns strictly increasing top to bottom.
- Longer columns to the left.

Schensted's algorithm To insert $a \in \mathcal{A}$ into a tableau:

1. If adding a to the end of the top row gives a tableau, this is the result.
2. Otherwise, let b the leftmost symbol of the top row such that $b > a$. Replace b with a ('bumping b').
3. Recursively insert b into the tableau formed by all lower rows.

For a word $u = u_1 u_2 \cdots u_k \in \mathcal{A}^*$:

- Start with an empty tableau and insert u_1, then u_2, ..., finally u_k.
- Call the resulting tableau $P_{\text{plac}}(u)$. For example, $P_{\text{plac}}(2531613443)$ is the tableau above.

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The plactic monoid

Schensted's algorithm computes a tableau $\mathsf{P}_{\mathsf{plac}}(u)$ for $u \in \mathcal{A}^*$.

Define $u \equiv_{\mathsf{plac}} v \iff \mathsf{P}_{\mathsf{plac}}(u) = \mathsf{P}_{\mathsf{plac}}(v)$.

Theorem (Knuth 1970)

The relation $\equiv_{\mathsf{plac}}$ is a congruence on $\mathcal{A}^*$.

- $\mathsf{plac} = \mathcal{A}^*/\equiv_{\mathsf{plac}}$ is the plactic monoid.
- $\mathsf{plac}_n = \mathcal{A}_n^*/\equiv_{\mathsf{plac}}$ is the plactic monoid of rank n.

Clearly,

$$\mathsf{plac}_1 \hookrightarrow \mathsf{plac}_2 \hookrightarrow \ldots \hookrightarrow plac_n \hookrightarrow \mathsf{plac}_{n+1} \hookrightarrow \ldots \hookrightarrow \mathsf{plac}.$$

and

$$\mathsf{plac} = \bigcup_{n \in \mathbb{N}} \mathsf{plac}_n.$$

Identities

- An identity is a formal equality $u = v$, where $u, v \in X^*$.
- A monoid M satisfies $u = v$ if substituting any element of M for each symbol in X gives an equality that holds in M.

For example,

- Any commutative monoid satisfies $xy = yx$.
- Any nilpotent group of class 2 satisfies $xyzyx = yxzxy$ [Neumann & Taylor 1963].

An identity is trivial if u and v are the same word; otherwise non-trivial.

- $xy = xy$ is trivial.
- $xy = yx$ and $xyzyx = yxzxy$ are non-trivial.

Questions

- Does plac satisfy a non-trivial identity?
- Does each plac_n satisfy a non-trivial identity?

Chinese monoid

- The Chinese monoid is also defined by an insertion algorithm.
- The Chinese monoid is related to the plactic monoid by its growth type.
- plac_2 is isomorphic to the Chinese monoid of rank 2.

Proposition (Jaszuńska & Okniński)

The Chinese monoid embeds into a direct product of copies of the bicyclic monoid and the infinite cyclic group.

Proposition (Adian)

The bicyclic monoid satisfies $xyyxxyxyyx = xyyxyxxyyx$ ('Adian's identity').

Corollary

The Chinese monoid satisfies Adian's identity.

Corollary

plac_2 satisfies Adian's identity.

Identities for plac_3

Adian's identity $xyyxxyxyyx = xyyxyxxyyx$

Proposition (Kubat & Okniński)

plac_3 satisfies $pqqpqp = pqpqqp$, where p and q are the left and right sides of Adian's identity. plac_3 does not satisfy Adian's identity.

- Use detailed calculations using normal forms.

Proposition (Izhakian)

plac_3 satisfies $xyxy^2x^2yxyxyxy^2x^2y = xyxy^2x^2yyxxyxy^2x^2y$.

- Use a (complicated) representation in the monoid of 3×3 upper-triangular tropical matrices.

Proposition (C., Klein, Kubat, Malheiro, Okniński)

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Identities for plac and plac_n

Theorem (C., Klein, Kubat, Malheiro, Okniński)

plac does not satisfy any non-trivial identity.

Proposition (CKKMO)

plac_n does not satisfy any non-trivial identity of length less than or equal to n.

Theorem (Schensted 1961)

$$\text{Number of columns in } P_{\text{plac}}(u) = \begin{cases}\text{Length of the longest weakly}\\ \text{increasing subsequence of } u;\end{cases}$$

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Proof that plac_n satisfies no identity of length at most n

Suppose plac_n satisfies $u(x, y) = v(x, y)$ of length n.

Assume

$$u = u_1 \cdots u_{j-1} x u_{j+1} \cdots u_n$$
$$v = v_1 \cdots v_{j-1} y v_{j+1} \cdots v_n$$

Let $s = 12 \cdots n \in \mathcal{A}_n^*$,
$t = 12 \cdots (n-j)(n-j+2) \cdots n \in \mathcal{A}_n^*$ (miss out $n-j+1$).

So the tableaux $\mathrm{P}_{\mathrm{plac}}\big(u(s,t)\big)$ and $\mathrm{P}_{\mathrm{plac}}\big(v(s,t)\big)$ are equal.

Longest decreasing subsequences:

In $u(s,t)$: $\quad u_1 \quad u_2 \quad \cdots \quad u_{j-1} \quad x \quad u_{j+1} \quad \cdots \quad u_{n-1} \quad u_n$

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$$

‘Plactic-like’ monoids

A family of monoids whose elements can be viewed as combinatorial objects:

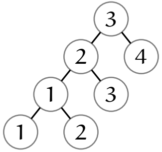
Plactic monoid
Young tableaux

1	1	2	3
2	3	4	
3			

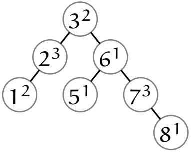
Hypoplactic monoid
Quasi-ribbon tableaux

1	1				
	2	2	3	3	3
					4

Sylvester monoid
Binary search trees



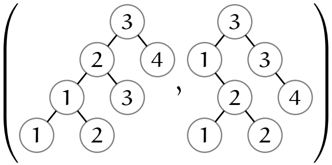
Taiga monoid
BSTs with multiplicities



Stalactite monoid
Stalactite tableaux

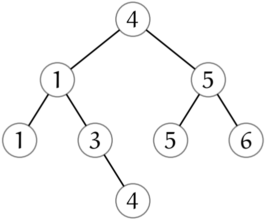
1	2	4	3
1	2		3
	2		3

Baxter monoid
Pairs of twin binary search trees



Binary search trees and leaf insertion

Binary search tree (BST):



To insert a into a BST T:

- Add a as a leaf node in the unique position that yields a BST.

For a word $u = u_k u_{k-1} \cdots u_1 \in \mathcal{A}^*$.

- Start with an empty BST and insert u_1, then u_2, ..., finally u_k.
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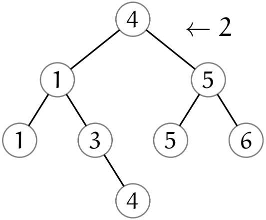

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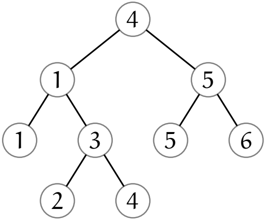

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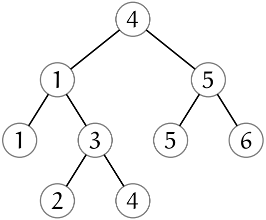

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The sylvester monoid

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Theorem (Hivert et al. 2005)

The relation $\equiv_{\mathsf{sylv}}$ is a congruence on $\mathcal{A}^*$.

- $\mathsf{sylv} = \mathcal{A}^*/{\equiv_{\mathsf{sylv}}}$ is the sylvester monoid.
- $\mathsf{sylv}_n = \mathcal{A}_n^*/{\equiv_{\mathsf{sylv}}}$ is the sylvester monoid of rank n.

Theorem (C., Malheiro)

sylv satisfies the identity $xyxy = yxxy$.
This is the unique shortest non-trivial identity satisfied by sylv.

- 'unique' up to renaming variables and swapping the two sides.

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- Want to show that $\mathsf{P}_{\mathsf{sylv}}(stst) = \mathsf{P}_{\mathsf{sylv}}(tsst)$ for all $s, t \in \mathcal{A}^*$.
- Suffices to prove that $\mathsf{P}_{\mathsf{sylv}}(pr) = \mathsf{P}_{\mathsf{sylv}}(qr)$ for all $p, q, r \in \mathcal{A}^*$, where p, q, r contain the same number of each symbol.

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$\mathsf{P}_{\mathsf{sylv}}(22853671 \quad 3761822{\color{red}5}) =$

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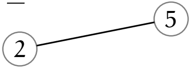

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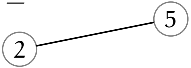

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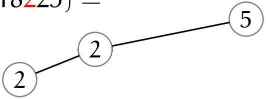

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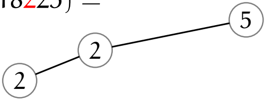

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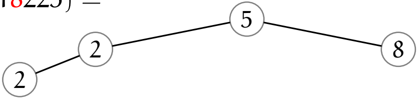

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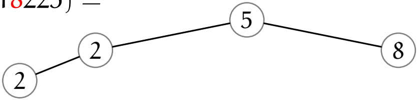

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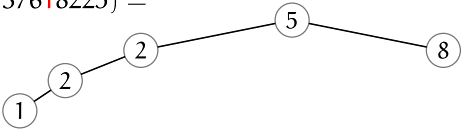

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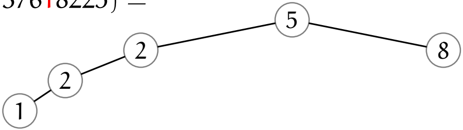

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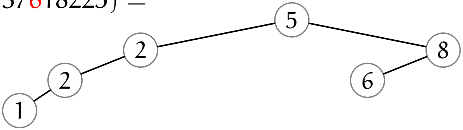

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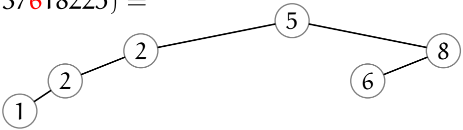

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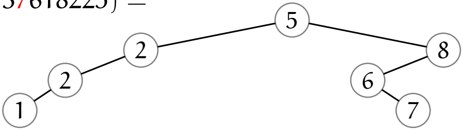

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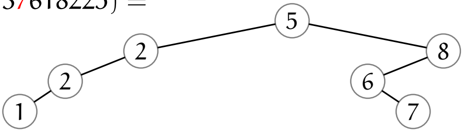

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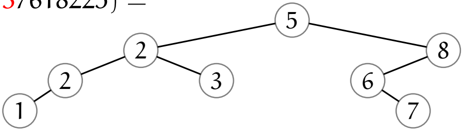

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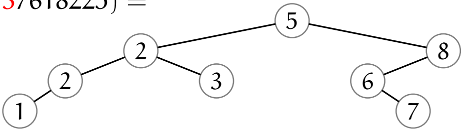

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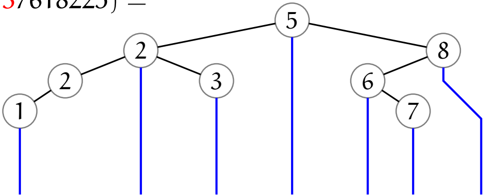

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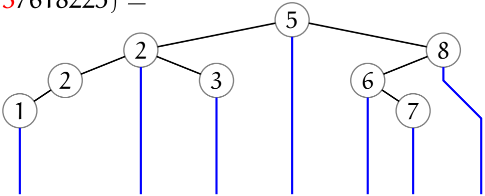

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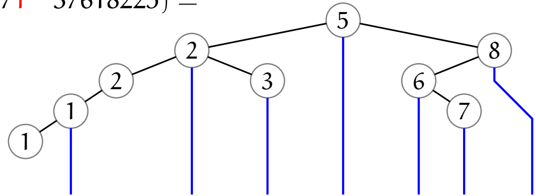

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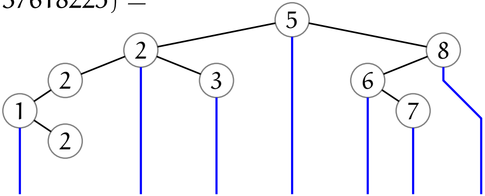

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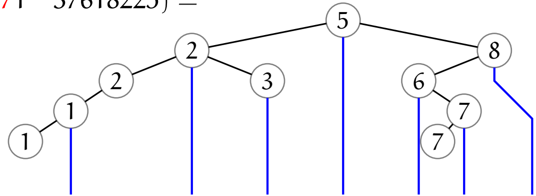

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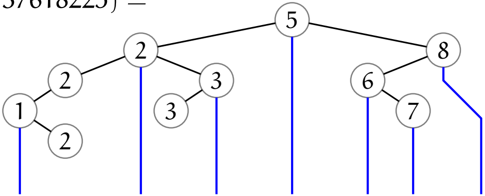

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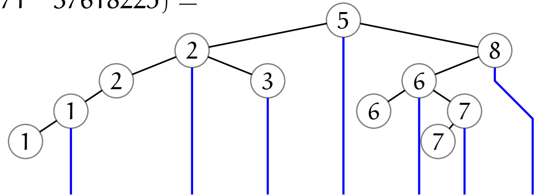

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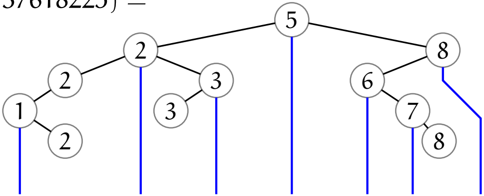

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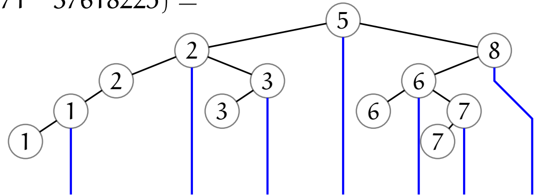

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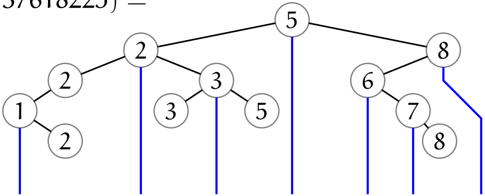

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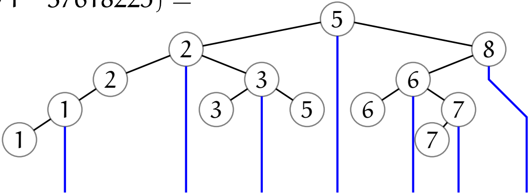

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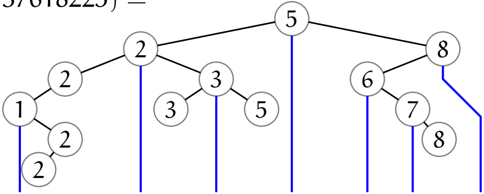

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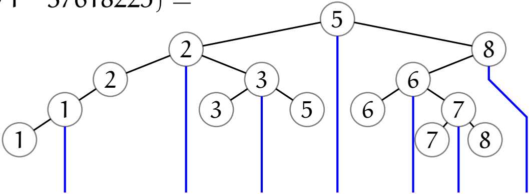

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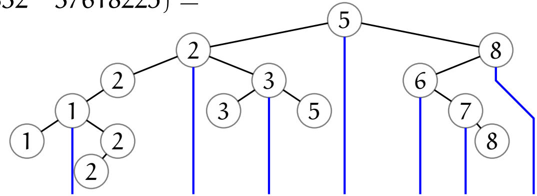

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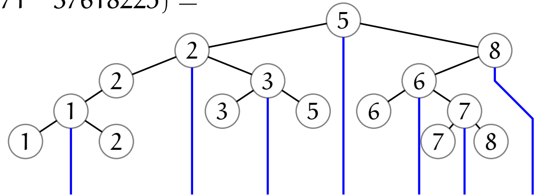

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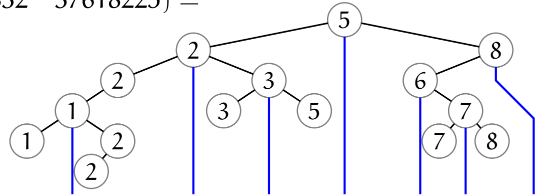

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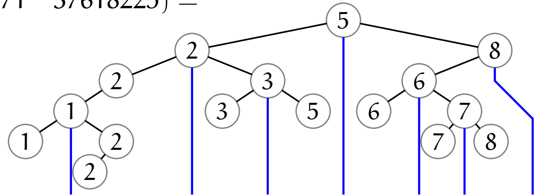

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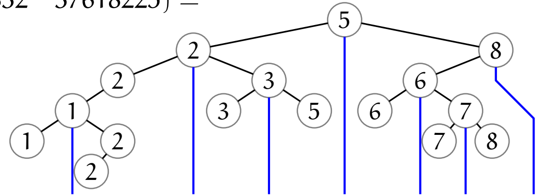

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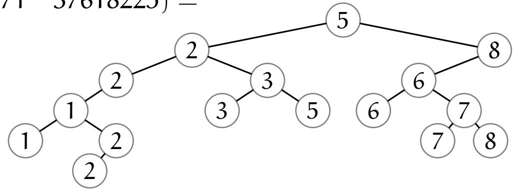

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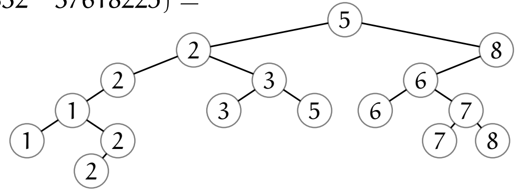

Quasi-ribbon tableaux & insertion

Quasi-ribbon tableau (QRT):

1	2			
	3	4	4	
			5	7

To insert a symbol a into a quasi-ribbon tableau T:

- Break the tableau two parts: $T_{\leqslant}$ is up to and including the bottom-right-most symbol r such that $r \leqslant a$; the remainder is $T_{>}$.
- Add a to the right of r.
- Attach $T_{>}$ to the bottom of a.

For a word $u = u_1 u_2 \cdots u_n \in \mathcal{A}^*$.

- Start with an empty QRT and insert u_1, then u_2, …, finally u_n.
- Call the resulting QRT $P_{\text{hypo}}(u)$. For example, $P_{\text{hypo}}(15344723)$is the QRT above.

Lemma

If $i < j$ are symbols in u and there is no k in u with $i < k < j$, then:

$$\begin{array}{c} i \text{ and } j \text{ are on the} \\ \text{different rows of } P_{\text{hypo}}(u) \end{array} \iff \begin{array}{l} \text{In } u\text{, some symbol } i \text{ is to the} \\ \text{right of some symbol } j \end{array}$$

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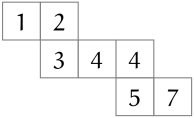

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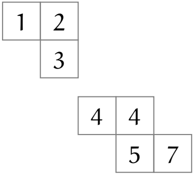

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Quasi-ribbon tableaux & insertion

Quasi-ribbon tableau (QRT):

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\hline
1 & 2 & \multicolumn{3}{c}{} \\
\hline
\multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c|}{3} & & \\
\cline{2-3}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{|c|}{4} & \\
\cline{3-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 5 & \multicolumn{1}{|c|}{7} \\
\cline{4-5}
\end{array}$$

To insert a symbol a into a quasi-ribbon tableau T:

- Break the tableau two parts: $T_{\leqslant}$ is up to and including the bottom-right-most symbol r such that $r \leqslant a$; the remainder is $T_{>}$.
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For a word $u = u_1 u_2 \cdots u_n \in \mathcal{A}^*$.

- Start with an empty QRT and insert u_1, then u_2, ..., finally u_n.
- Call the resulting QRT $P_{\text{hypo}}(u)$. For example, $P_{\text{hypo}}(15344723)$is the QRT above.

Lemma

If $i < j$ are symbols in u and there is no k in u with $i < k < j$, then:

i and j are on the different rows of $P_{\text{hypo}}(u)$ $\iff$ In u, some symbol i is to the right of some symbol j

Quasi-ribbon tableaux & insertion

Quasi-ribbon tableau (QRT):

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The hypoplactic monoid

Define $u \equiv_{\mathsf{hypo}} v \iff \mathsf{P}_{\mathsf{hypo}}(u) = \mathsf{P}_{\mathsf{hypo}}(v)$.

Theorem (Novelli 2000)

The relation $\equiv_{\mathsf{hypo}}$ is a congruence on A^*.

- $\mathsf{hypo} = \mathcal{A}^*/{\equiv_{\mathsf{hypo}}}$ is the hypoplactic monoid.
- $\mathsf{hypo}_n = \mathcal{A}_n^*/{\equiv_{\mathsf{hypo}}}$ is the hypoplactic monoid of rank n.

Theorem (C., Malheiro)

hypo satisfies the identities

$$xyxy = xyyx = yxxy = yxyx;$$
$$xxyx = xyxx.$$

These are the unique shortest non-trivial identities satisfied by hypo.

- A QRT is determined by the number of each symbol it contains and which symbols are on the same rows.
- These are the length-4 identities where the two sides preserve these properties.

Summary table

Monoid	*Symbol*	*Identity*	*In rank* n
Plactic	plac	None	?
Hypoplactic	hypo	$xyxy = yxyx$	Y
Sylvester	sylv	$xyxy = yxxy$	Y
Baxter	baxt	$yxxyxy = yxyxxy$	Y
Stalactic	stal	$xyx = yxx$	Y
Taiga	taig	$xyx = yxx$	Y
Left patience sorting	lPS	None	N
Right patience sorting	rPS	None	Y

Question

Does plac_n satisfy a non-trivial identity for $n \geqslant 4$?

- Conjectured hierarchy of identities for plac_n, length $2 \times 5^{n-1}$.
- Lots of random examples checked in plac_4 using Sage.