Hyperplane arrangements: between Shi and Ish

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For an integer $n \geq 3$, and integers $i, j$ such that $1 \leq i < j \leq n$,

\[
C_{ij} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j \right\},
\]
\[
S_{ij} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j + 1 \right\},
\]
\[
I_{ij} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = x_j + i \right\}.
\]

$n$-dimensional... 

... Coxeter arrangement: $\text{Cox}_n = \bigcup_{1 \leq i < j \leq n} C_{ij}$,

... Shi arrangement: $\text{Shi}_n = \text{Cox}_n \cup \bigcup_{1 \leq i < j \leq n} S_{ij}$, (Jian-Yi Shi, 1986)

... Ish arrangement: $\text{Ish}_n = \text{Cox}_n \cup \bigcup_{1 \leq i < j \leq n} I_{ij}$, (Drew Armstrong, 2010)
Hyperplane arrangements: between Shi and Ish

Introduction

Figure: Shi_3 and Ish_3
[n] := \{1, \ldots, n\}

(k, n) := \{k + 1, \ldots, n - 1\}, \quad \text{for } 1 \leq k < n.

For \(X \subseteq (1, n)\),

\[ A^X := \text{Cox}_n \cup \bigcup_{i \in X} S_{ij} \cup \bigcup_{i \in [n] \setminus X} l_{ij}. \]

Then

\[ \text{Shi}_n = A^{(1,n)} \]

and

\[ \text{Ish}_n = A^\emptyset = A^{(n-1,n)}. \]
### Introduction

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<th>Shi$_4 = \mathcal{A}^{(1,4)}$</th>
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Characteristic polynomials

**Theorem (Athanasiadis, 1996)**
\[ \chi(\text{Shi}_n, q) = \chi(\mathcal{A}^{(1,n)}, q) = q(q - n)^{n-1}. \]

**Theorem (Ardila, 2015)**
\[ \chi(\text{Ish}_n, q) = \chi(\mathcal{A}^{(n-1,n)}, q) = q(q - n)^{n-1}. \]

**Theorem**

*For every integer $1 \leq k < n,*
\[ \chi(\mathcal{A}^{(k,n)}, q) = q(q - n)^{n-1}. \]
Corollary

For $1 \leq k < n$, the number of regions of $A^{(k,n)}$ is
$r(A^{(k,n)}) = (-1)^n \chi(-1) = (n + 1)^{n-1},$
and the number of relatively bounded regions of $A^{(k,n)}$ is
$b(A^{(k,n)}) = (-1)^{n-1} \chi(1) = (n - 1)^{n-1}.$

In general, if $X \subseteq (1, n)$, then
$\chi(A^X, q) \neq q(q - n)^{n-1},$
$r(A^X) \neq (n + 1)^{n-1}$ and
$b(A^X) \neq (n - 1)^{n-1}.$
The following labeling...

- is similar to the labeling introduced by Pak and Stanley for the Shi arrangement,
- was studied in a general setting that covers our arrangements by Mazin.

$\mathbf{e}_i \in \{0, 1, \ldots, n-1\}^n$ such that all coordinates are 0 except the $i$th coordinate, which is 1

$\mathcal{R}_0 =$ region defined by $x_n + 1 > x_1 > x_2 > \cdots > x_n$
(limited by $l_{1n} = S_{1n}$ and by $C_{ij}$ and $I_{ij}$ for $1 \leq j < n$)

Pak-Stanley labeling:

$\ell(\mathcal{R}_0) = (0, \ldots, 0)$

Given two regions, $\mathcal{R}_1$ and $\mathcal{R}_2$, separated by $H$ such that $\mathcal{R}_0$ and $\mathcal{R}_1$ are on the same side of $H$, let

$$
\ell(\mathcal{R}_2) = \ell(\mathcal{R}_1) + \begin{cases} 
\mathbf{e}_i, & \text{if } H \in C_{ij} \\
\mathbf{e}_j, & \text{if } H \in S_{ij} \cup I_{ij}
\end{cases}
$$
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The Pak-Stanley labeling

Figure: Pak-Stanley labelings of Shi₃ and Ish₃
For $X \subseteq (1, n)$, let $G^X = (V, A^X)$ be the directed multigraph such that

- $V = [n]$
- $A^X$ is the multiset with $|A^X|$ elements formed by adding, for each $H$, a new arc $a \in A^X$, as follows. If $H = C_{ij}$ for some $1 \leq i < j \leq n$, then $a = (i, j)$; if $H = S_{ij}$, then $a = (j, i)$; if $H = I_{ij}$, then $a = (j, 1)$. Note that an arc of form $(j, 1)$ may occur more than once in the multiset $A^X$. 

Hyperplane arrangements: between Shi and Ish

The Pak- Stanley labeling

Figure: Directed multi-graphs associated with Shi$_3$ ($G^{(1,3)}$) and Ish$_3$ ($G^\varnothing$)
Definition (Mazin, 2015)

Let $G = (V, A)$ be a (finite) directed connected multigraph without loops, where $V = [n]$ for some $n \in \mathbb{N}$. 

$\mathcal{P} : [n] \to \mathbb{N}_0$ is a $G$-parking function iff 

\[ \forall \emptyset \neq I \subseteq [n] \; \exists i \in I : \text{the number of elements } j \notin I \text{ such that } (i, j) \in A, \text{ counted with multiplicity, is } \geq \mathcal{P}(i). \]

Theorem (Mazin, 2015)

*The set of labels obtained above is the set of $\mathcal{G}^X$-parking functions.*
For $A^{(1,n)} = \text{Shi}_n$, $G^{(1,n)}$-parking functions are the parking functions, $PF_n$.

For $A^\emptyset = \text{Ish}_n$, $G^\emptyset$-parking functions are the Ish-parking functions, $\text{IPF}_n$.

$a : [n] \to \{0, 1, \ldots, n - 1\}$ is a parking function iff

$$|a^{-1}(\{0, 1, \ldots, i\})| > i \quad (i = 0, 1, \ldots, n - 1).$$
Let $\overline{G} = (\overline{V}, \overline{A})$ where $\overline{V} = \{0\} \cup V$ and

$$ \overline{A} = \{(0, i) | i \in [n]\} \cup \{(j, i) | (i, j) \in A\} $$

and define, for each $v \in \overline{V}$, the list $\mathcal{N}(v)$ by ordering in some fixed way the multiset

$$ \{ i \in [n] | (\nu, i) \in \overline{A} \} = \begin{cases} [n], & \text{if } \nu = 0 \\ \{ i \in [n] | (i, \nu) \in A\}, & \text{if } \nu \neq 0 \end{cases} $$

We extend the DFS-burning algorithm by Perkinson, Yang and Yu (2017) and part of their results to directed multi-graphs $G = (V, A)$. 
Algorithm 1 DFS-burning algorithm (adapted).

ALGORITHM

Input: $a: [n] \to \mathbb{N}_0$

1: burnt_vertices = \{0\}
2: dampened_edges = \{}
3: tree_edges = \{}
4: execute DFS_FROM(0)

Output: burnt_vertices, tree_edges and dampened_edges

AUXILIARY FUNCTION

5: function DFS_FROM(i)
6: foreach $j$ in neighbors(i) do
7: if $j \notin$ burnt_vertices then
8: if $a(j) = 0$ then
9: append $(i, j)$ to tree_edges
10: append $j$ to burnt_vertices
11: DFS_FROM(j)
12: else
13: append $(i, j)$ to dampened_edges
14: $a(j) = a(j) - 1$
Theorem
Given a directed multigraph $G$ on $[n]$ and a function $P : [n] \rightarrow \mathbb{N}_0$, $P$ fits $G$ if and only if $P$ is a $G$-parking function.

Theorem
The $G$-parking functions are in bijection with the spanning arborescences (the directed rooted trees with edges pointing away from the root) of $G$ that are rooted in 0.

Theorem
For every natural $n \geq 3$ and for every $X \subseteq (1, n)$, the Pak-Stanley labeling considered above defines a bijection between the regions of $A^X$ and the $G^X$-parking functions.
Definition
Let $a = (a_1, \ldots, a_n) \in \mathbb{N}_0^n$. The reverse center of $a$, $\tilde{Z}(a)$, is the largest subset $X = \{x_1, \ldots, x_\ell\}$ of $[n]$ such that $n \geq x_1 > \cdots > x_\ell \geq 1$ and $a_{x_i} < i$ for every $i \in [\ell]$.

Theorem
A function $a: [n] \rightarrow \mathbb{N}_0$ is an Ish-parking function if and only if 1 belongs to the reverse center of $\mathcal{P}$. 

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The number of Ish-parking functions with reverse center of a given length

\[
\begin{align*}
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|\text{IPF}_n^k| &= k! \sum_{j_1+j_2+\cdots+j_k=n-k} (n-1)^{j_1} (n-2)^{j_2} \cdots (n-k)^{j_k} \\
&= k \times \left| \left\{ f : [n-1] \to [n-1] \mid [k-1] \subseteq f([n-1]) \right\} \right| \\
&= k \times \sum_{j=0}^{r-1} (-1)^j \binom{k-1}{j} (n-1-j)^{n-1}
\end{align*}
\end{align*}
\]
Hyperplane arrangements: between Shi and Ish

The number of Ish-parking functions with reverse center of a given length

Conjecture: (Verified up to $n = 8$.) For every $n \in \mathbb{N}$, $n \geq 3$, the number of $G^X$-parking functions with reverse center of a given length does not depend on $X \subseteq (1, n)$. 
Thank you for your attention!