Hyperplane arrangements: between Shi and Ish

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For an integer $n \geq 3$, and integers i, j such that $1 \leq i < j \leq n$,

$$C_{ij} = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = x_j\right\},$$
$$S_{ij} = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = x_j + 1\right\},$$
$$I_{ij} = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = x_j + i\right\}.$$

n-dimensional…

…Coxeter arrangement: $\mathrm{Cox}_n = \bigcup_{1 \leq i < j \leq n} C_{ij}$,

…Shi arrangement: $\mathrm{Shi}_n = \mathrm{Cox}_n \cup \bigcup_{1 \leq i < j \leq n} S_{ij}$, (Jian-Yi Shi, 1986)

…Ish arrangement: $\mathrm{Ish}_n = \mathrm{Cox}_n \cup \bigcup_{1 \leq i < j \leq n} I_{ij}$, (Drew Armstrong, 2010)

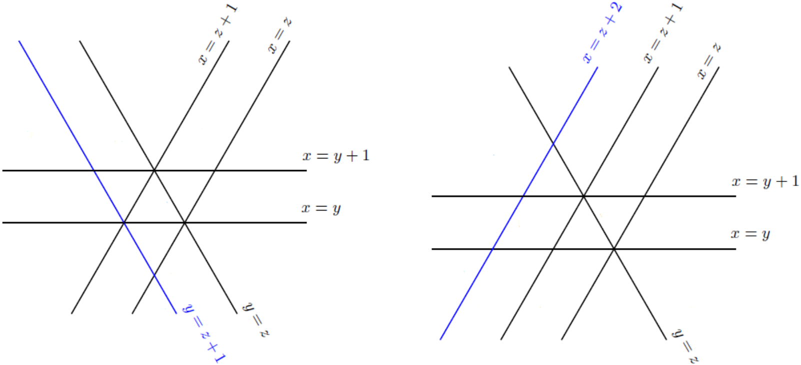

Figure: Shi_3 and Ish_3

$$[n] := \{1, \ldots, n\}$$
$$(k, n) := \{k+1, \ldots, n-1\}, \quad \text{for } 1 \leq k < n.$$

For $X \subseteq (1, n)$,

$$\mathcal{A}^X := \mathrm{Cox}_n \cup \bigcup_{\substack{i \in X \\ i < j \leq n}} S_{ij} \cup \bigcup_{\substack{i \in [n] \setminus X \\ i < j \leq n}} I_{ij} \,.$$

Then

$$\mathrm{Shi}_n = \mathcal{A}^{(1,n)}$$

and

$$\mathrm{Ish}_n = \mathcal{A}^{\varnothing} = \mathcal{A}^{(n-1,n)}.$$

$\mathrm{Shi}_4 = \mathcal{A}^{(1,4)}$	$\mathcal{A}^{(2,4)}$	$\mathrm{Ish}_4 = \mathcal{A}^{(3,4)}$
$x_1 = x_2$	$x_1 = x_2$	$x_1 = x_2$
$x_1 = x_3$	$x_1 = x_3$	$x_1 = x_3$
$x_1 = x_4$	$x_1 = x_4$	$x_1 = x_4$
$x_2 = x_3$	$x_2 = x_3$	$x_2 = x_3$
$x_2 = x_4$	$x_2 = x_4$	$x_2 = x_4$
$x_3 = x_4$	$x_3 = x_4$	$x_3 = x_4$
$x_1 = x_2 + 1$	$x_1 = x_2 + 1$	$x_1 = x_2 + 1$
$x_1 = x_3 + 1$	$x_1 = x_3 + 1$	$x_1 = x_3 + 1$
$x_1 = x_4 + 1$	$x_1 = x_4 + 1$	$x_1 = x_4 + 1$
$x_2 = x_3 + 1$	$x_1 = x_3 + 2$	$x_1 = x_3 + 2$
$x_2 = x_4 + 1$	$x_1 = x_4 + 2$	$x_1 = x_4 + 2$
$x_3 = x_4 + 1$	$x_3 = x_4 + 1$	$x_1 = x_4 + 3$

Characteristic polynomials

Theorem (Athanasiadis, 1996)

$\chi(\mathrm{Shi}_n, q) = \chi(\mathcal{A}^{(1,n)}, q) = q(q-n)^{n-1}$.

Theorem (Ardila, 2015)

$\chi(\mathrm{Ish}_n, q) = \chi(\mathcal{A}^{(n-1,n)}, q) = q(q-n)^{n-1}$.

Theorem

For every integer $1 \leq k < n$,
$\chi(\mathcal{A}^{(k,n)}, q) = q(q-n)^{n-1}$.

Corollary

For $1 \leq k < n$, the number of regions of $\mathcal{A}^{(k,n)}$ is
$r(\mathcal{A}^{(k,n)}) = (-1)^n \chi(-1) = (n+1)^{n-1}$,
and the number of relatively bounded regions of $\mathcal{A}^{(k,n)}$ is
$b(\mathcal{A}^{(k,n)}) = (-1)^{n-1} \chi(1) = (n-1)^{n-1}$.

In general, if $X \subseteq (1, n)$, then
$\chi(\mathcal{A}^X, q) \neq q(q-n)^{n-1}$,
$r(\mathcal{A}^X) \neq (n+1)^{n-1}$ and
$b(\mathcal{A}^X) \neq (n-1)^{n-1}$.

The following labeling. . .

- is similar to the labeling introduced by Pak and Stanley for the Shi arrangement,
- was studied in a general setting that covers our arrangements by Mazin.

$\mathbf{e}_i \in \{0, 1, \ldots, n-1\}^n$ such that all coordinates are 0 except the ith coordinate, which is 1

$\mathcal{R}_0 =$ region defined by $x_n + 1 > x_1 > x_2 > \cdots > x_n$
(limited by $I_{1n} = S_{1n}$ and by $C_{j\,j+1}$ for $1 \leq j < n$)

Pak-Stanley labeling:

$\ell(\mathcal{R}_0) = (0, \ldots, 0)$
Given two regions, $\mathcal{R}_1$ and $\mathcal{R}_2$, separated by H such that $\mathcal{R}_0$ and $\mathcal{R}_1$ are on the same side of H, let

$$\ell(\mathcal{R}_2) = \ell(\mathcal{R}_1) + \begin{cases} \mathbf{e}_i, & \text{if } H \in C_{ij} \\ \mathbf{e}_j, & \text{if } H \in S_{ij} \cup I_{ij} \end{cases}$$

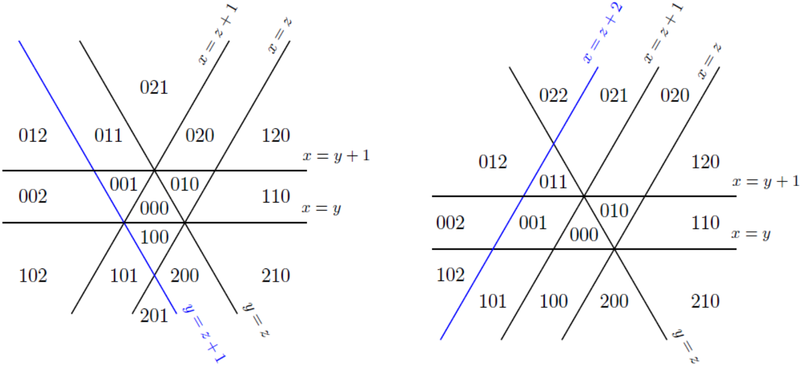

Figure: Pak-Stanley labelings of Shi_3 and Ish_3

For $X \subseteq (1, n)$, let $\mathcal{G}^X = (V, A^X)$ be the directed multigraph such that

- $V = [n]$
- A^X is the multiset with $|\mathcal{A}^X|$ elements formed by adding, for each H, a new arc $a \in A^X$, as follows. If $H = C_{ij}$ for some $1 \le i < j \le n$, then $a = (i, j)$; if $H = S_{ij}$, then $a = (j, i)$; if $H = I_{ij}$, then $a = (j, 1)$.

Note that an arc of form $(j, 1)$ may occur more than once in the multiset A^X.

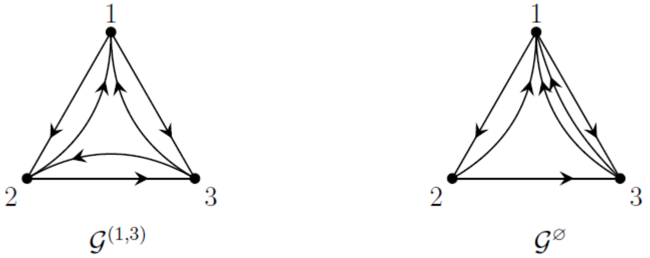

Figure: Directed multi-graphs associated with Shi_3 ($\mathcal{G}^{(1,3)}$) and Ish_3 ($\mathcal{G}^{\varnothing}$)

Definition (Mazin, 2015)

Let $\mathcal{G} = (V, A)$ be a (finite) directed connected multigraph without loops, where $V = [n]$ for some $n \in \mathbb{N}$.
$\mathcal{P} \colon [n] \to \mathbb{N}_0$ is a *$\mathcal{G}$-parking function* iff
$\forall \varnothing \neq I \subseteq [n]\ \exists i \in I$: the number of elements $j \notin I$ such that $(i, j) \in A$, counted with multiplicity, is $\geq \mathcal{P}(i)$.

Theorem (Mazin, 2015)

The set of labels obtained above is the set of $\mathcal{G}^X$-parking functions.

- For $\mathcal{A}^{(1,n)} = \mathrm{Shi}_n$, $\mathcal{G}^{(1,n)}$*-parking functions* are the *parking functions*, PF_n.
- For $\mathcal{A}^{\varnothing} = \mathrm{Ish}_n$, $\mathcal{G}^{\varnothing}$*-parking functions* are the *Ish-parking functions*, IPF_n.

$\mathbf{a}\colon [n] \to \{0, 1, \dots, n-1\}$ is a *parking function* iff

$$|\mathbf{a}^{-1}(\{0, 1, \dots, i\})| > i \qquad (i = 0, 1, \dots, n-1)\,.$$

Let $\overline{\mathcal{G}} = (\overline{V}, \overline{A})$ where $\overline{V} = \{0\} \cup V$ and

$$\overline{A} = \{(0,i) \mid i \in [n]\} \cup \{(j,i) \mid (i,j) \in A\}$$

and define, for each $v \in \overline{V}$, the list $\mathcal{N}(v)$ by ordering in some fixed way the *multiset*

$$\{i \in [n] \mid (v,i) \in \overline{A}\} = \begin{cases} [n], & \text{if } v = 0 \\ \{i \in [n] \mid (i,v) \in A\}, & \text{if } v \neq 0 \end{cases}$$

We extend the DFS-burning algorithm by Perkinson, Yang and Yu (2017) and part of their results to directed multi-graphs $\mathcal{G} = (V, A)$.

Algorithm 1 DFS-burning algorithm (adapted).

ALGORITHM

```
  Input: a: [n] → ℕ_0
  burnt_vertices = {0}
  dampened_edges = { }
  tree_edges = { }
  execute DFS_FROM(0)
  Output: burnt_vertices, tree_edges and dampened_edges
```

AUXILIARY FUNCTION

```
function DFS_FROM(i)
    foreach j in neighbors(i) do
        if j ∉ burnt_vertices then
            if a(j) = 0 then
                append (i, j) to tree_edges
                append j to burnt_vertices
                DFS_FROM(j)
            else
                append (i, j) to dampened_edges
                a(j) = a(j) − 1
```

Theorem

Given a directed multigraph $\mathcal{G}$ on $[n]$ and a function $\mathcal{P}\colon [n] \to \mathbb{N}_0$, $\mathcal{P}$ fits $\mathcal{G}$ if and only if $\mathcal{P}$ is a $\mathcal{G}$-parking function.

Theorem

The $\mathcal{G}$-parking functions are in bijection with the spanning arborescences (the directed rooted trees with edges pointing away from the root) of $\mathcal{G}$ that are rooted in 0.

Theorem

For every natural $n \geq 3$ and for every $X \subseteq (1, n)$, the Pak-Stanley labeling considered above defines a bijection between the regions of $\mathcal{A}^X$ and the $\mathcal{G}^X$-parking functions.

Definition

Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}_0^n$. The *reverse center of* $\mathbf{a}$, $\widetilde{Z}(\mathbf{a})$, is the largest subset $X = \{x_1, \dots, x_\ell\}$ of $[n]$ such that $n \geq x_1 > \cdots > x_\ell \geq 1$ and $a_{x_i} < i$ for every $i \in [\ell]$.

Theorem

A function $\mathbf{a} \colon [n] \to \mathbb{N}_0$ *is an Ish-parking function if and only if* 1 *belongs to the reverse center of* $\mathcal{P}$.

$$\begin{array}{cccccccc}
\boxed{a_{i_k} \le k-1} & \neq \left\{\begin{array}{l} 0 \\ 1 \\ \vdots \\ k-1 \end{array}\right. & & \neq \left\{\begin{array}{l} 0 \\ 1 \\ 2 \end{array}\right. & \boxed{a_{i_2} \le 1} & \neq \left\{\begin{array}{l} 0 \\ 1 \end{array}\right. & \boxed{a_{i_1} \le 0} & \neq 0 \\
\underbrace{\qquad\quad}_{i_k=1} & \underline{\qquad\qquad} & \cdots & \underline{\qquad\qquad} & \underbrace{\qquad\quad}_{i_2} & \underline{\qquad\qquad} & \underbrace{\qquad\quad}_{i_1} & \underline{\qquad\qquad}
\end{array}$$

$$\begin{aligned}
|\mathrm{IPF}_n^k| &= k! \sum_{j_1+j_2+\cdots+j_k=n-k} (n-1)^{j_1}(n-2)^{j_2}\cdots(n-k)^{j_k} \\
&= k \times \left|\left\{ f\colon [n-1] \to [n-1] \mid [k-1] \subseteq f([n-1]) \right\}\right| \\
&= k \times \sum_{j=0}^{r-1} (-1)^j \binom{k-1}{j} (n-1-j)^{n-1}
\end{aligned}$$

Conjecture: (Verified up to $n = 8$.) For every $n \in \mathbb{N}$, $n \geq 3$, the number of $\mathcal{G}^X$-parking functions with reverse center of a given length does not depend on $X \subseteq (1, n)$.

Thank you for your attention!