On the combinatorics of the symplectic group

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(joint with Bruce Westbury and with Stephan Pfannerer and Bruce Westbury)

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Invariants
Background: Invariants

- $G$ a group of matrices - $\text{GL}_n$, $\text{O}_n$, $\text{Sp}_{2n}$, $\mathfrak{S}_n$
- $\mathcal{V}$ a $G$-module - natural or adjoint representation

Goal: describe the ring of polynomial invariants

$\mathbb{C} [\mathcal{V}]^G = \{ f \text{ polynomial in the coordinates of } \mathcal{V} : \forall g \in G, v \in \mathcal{V} : f(gv) = f(v) \}.$

First Fundamental Theorem (FFT): explicit generating system
Second Fundamental Theorem (SFT): relations between generators
Background: Invariants

Equivalently (in characteristic 0), describe

\[ \text{Hom}_G \left( V^\otimes r, V^\otimes s \right) = \{ f \in \text{Hom} \left( V^\otimes r, V^\otimes s \right) : \right. \]
\[ \forall g \in G, v \in V^\otimes r : f(gv) = gf(v) \} \].

**FFT**: explicit generating system

**SFT**: linear relations between generators
Background: Ancestors

David Hilbert, ‘Über die Theorie der algebraischen Formen’, 1890.

14th problem: is the ring of invariants finitely generated?


first and second fundamental theorem for the classical groups and their natural representation

Richard Brauer, ‘On algebras which are connected with the semisimple continuous groups’, 1937.

combinatorial description of the invariants of the orthogonal and the symplectic group
Background: Ancestors

explicit bounds for finite groups

first and second fundamental theorem for the classical groups and their natural representation

Richard Brauer, ‘On algebras which are connected with the semisimple continuous groups’, 1937.
combinatorial description of the invariants of the orthogonal and the symplectic group
Background: Ancestors

explicit bounds for finite groups

**Hermann Weyl**, ‘The classical groups - their invariants and representations’, 1939.
first and second fundamental theorem for the classical groups and their natural representation

combinatorial description of the invariants of the symplectic group in low dimension
Background: Ancestors

explicit bounds for finite groups

Judith Braunsteiner, ‘A Sundaram bijection for the odd orthogonal groups’, 2017?.
combinatorial description of the invariants of the odd orthogonal group in low dimension

combinatorial description of the invariants of the symplectic group in low dimension
Background: classical results

- natural representation of $\text{GL}_n$ (Schur, 1901)  
  (a matrix $g$ acts on $v \in \mathbb{C}^n$ as $g \cdot v$)
- natural representation of $\text{O}_n$, $\text{Sp}_{2n}$ (Weyl, 1924)
- natural representation of $\mathfrak{S}_n$
- adjoint representation of $\text{GL}_n$  
  (a matrix $g$ acts on $v \in \text{Mat}_{n,n}(\mathbb{C})$ as $g \cdot v \cdot g^{-1}$)
Background: $\text{GL}_n$, natural representation (Schur, Weyl)

**FFT:** $\text{Hom}_{\text{GL}_n}(V \otimes r, V \otimes s) = 0$ for $r \neq s$, and the algebra homomorphism

\[ \text{ev}_n : \mathbb{C} \mathcal{S}_r \to \text{Hom}_{\text{GL}_n}(V \otimes r, V \otimes r) \]

\[ \text{ev}_n(\sigma) = v_1 \otimes \cdots \otimes v_r \mapsto v_{\sigma^{-1} 1} \otimes \cdots \otimes v_{\sigma^{-1} r} \]

is surjective.
Background: $\text{GL}_n$, natural representation (Schur, Weyl)

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ev_n(\sigma) = v_1 \otimes \cdots \otimes v_r \mapsto v_{\sigma^{-1}1} \otimes \cdots \otimes v_{\sigma^{-1}r}
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is surjective.

**SFT:** The kernel of $\ev_n$ is generated by the antisymmetriser
\[
E(n + 1) = \sum_{\pi \in S_{n+1}} \varepsilon(\pi)\pi.
\]
Background: $\text{GL}_n$, natural representation (Schur, Weyl)

**FFT:** $\text{Hom}_{\text{GL}_n} (V \otimes r, V \otimes s) = 0$ for $r \neq s$, and the algebra homomorphism

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**SFT:** The kernel of $\ev_n$ is generated by the antisymmetriser

$$E(n + 1) = \sum_{\pi \in \mathcal{S}_{n+1}} \varepsilon(\pi) \pi.$$

Alternatively:

$$\{ \ev_n(\pi) \mid \text{longest decreasing subsequence in } \pi \text{ has length at most } n \}$$
is a basis of $\text{Hom}_{\text{GL}_n} (V \otimes r, V \otimes r)$. 
Idea: algebraic combinatorics

\[
\text{algebra} \quad \longleftrightarrow \quad \text{combinatorics}
\]

\[
\text{Hom}_{GL_n} (V \otimes^r, V \otimes^r) \quad \longleftrightarrow \quad (r = 5, \ n \geq 2)
\]

composition \quad \longleftrightarrow \quad \text{stack}

tensor product \quad \longleftrightarrow \quad \text{draw side by side}

\[
(r = 5, \ n \geq 2)
\]

(modulo ker ev_n)
Let $\mathcal{B}_{r,s}$ be the set of perfect matchings of $1, \ldots, r + s$.

**FFT:** $\text{Hom}_{\text{Sp}_{2n}}(V^\otimes r, V^\otimes s) = 0$ when $r + s$ odd, and the evaluation functor

$$\text{ev}_n: \mathcal{B}_{r,s} \rightarrow \text{Hom}_{\text{Sp}_{2n}}(V^\otimes r, V^\otimes s)$$

is full (i.e., surjective on objects).

**SFT:** The kernel of $\text{ev}_n$ is generated by

$$E(n + 1) = \sum_{\pi \in \mathcal{B}_{n+1,n+1}} \pi.$$

Alternatively (R. & Westbury):

$$\{\text{ev}_n(\pi) \mid \pi \text{ is } (n + 1)\text{-noncrossing}\}$$

is a basis of $\text{Hom}_{\text{Sp}_{2n}}(V^\otimes r, V^\otimes s)$. 

---

**Sp}_{2n}, natural representation (Brauer, Weyl)**
Brauer’s category

$\mathcal{B}_{r,s}$ is the set of perfect matchings of 1, \ldots, $r + s$.

$r = 3$

$s = 5$
Brauer’s category

\( \mathcal{B}_{r,s} \) is the set of perfect matchings of 1, \ldots, \( r + s \).

\[
\begin{array}{c}
\text{identity} & \longleftrightarrow & \\
\text{composition} & \longleftrightarrow & \text{stack} \\
\text{tensor product} & \longleftrightarrow & \text{draw side by side}
\end{array}
\]

When stacking, loops may occur. Remove them, then multiply the result with \( \delta = -2^n \) for each loop.
Brauer’s category

\( \mathcal{B}_{r,s} \) is the set of perfect matchings of 1, \ldots, \( r + s \).

\( r = 3 \)
\( s = 5 \)

\( \mathbb{C} \mathcal{B}_{r,s} \) is the set of morphisms of ‘Brauer’s category’:

- identity \( \Leftarrow \rightarrow \)
- composition \( \Leftarrow \rightarrow \)
- tensor product \( \Leftarrow \rightarrow \)

When stacking, loops may occur. Remove them, then multiply the result with \( \delta = -2n \) for each loop:

\[ \delta^2 \]
The evaluation functor

- $b_1, \ldots, b_{2n}$ a basis of $V$,
- $\langle \ , \ \rangle$ a skew symmetric bilinear form,
- $b_1^*, \ldots, b_{2n}^*$ the dual basis, regarded as basis of $V$: $\langle b_i^*, b_j \rangle = \delta_{i,j}$.

Define

$$ev_n : \mathcal{B}_{r,s} \to \text{Hom}_{\text{Sp}_{2n}} (V^\otimes r, V^\otimes s)$$

by

$$ev_n (\bigotimes) = u \otimes v \mapsto -v \otimes u$$
$$ev_n (\bigcirc) = 1 \mapsto \sum_i b_i \otimes b_i^*$$
$$ev_n (\bigcirc) = u \otimes v \mapsto \langle u, v \rangle.$$
The evaluation functor

- \( b_1, \ldots, b_{2n} \) a basis of \( V \),
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  \( \langle b_i^*, b_j \rangle = \delta_{i,j} \).

Define

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ev_n : \mathbb{B}_{r,s} \to \text{Hom}_{\text{Sp}_{2n}} (V \otimes^r, V \otimes^s)
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by

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ev_n (\begin{array}{c}
\star
\end{array}) = u \otimes v \mapsto -v \otimes u
\]

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ev_n (\begin{array}{c}
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\]

\[
ev_n (\begin{array}{c}
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\end{array}) = u \otimes v \mapsto \langle u, v \rangle.
\]

(exercise: these are \( \text{Sp}_{2n} \)-invariants.)
\textbf{Sp}_{2n}, natural representation (Brauer, Weyl)

Let \( \mathcal{B}_{r,s} \) be the set of perfect matchings of \( 1, \ldots, r+s \).

\textbf{FFT:} \( \text{Hom}_{\text{Sp}_{2n}} ( V \otimes r, V \otimes s ) = 0 \) when \( r + s \) odd, and the evaluation functor

\[
\text{ev}_n : \mathbb{C} \mathcal{B}_{r,s} \rightarrow \text{Hom}_{\text{Sp}_{2n}} ( V \otimes r, V \otimes s )
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is full (i.e., surjective on objects).

\textbf{SFT:} The kernel of \( \text{ev}_n \) is generated by

\[
E(n + 1) = \sum_{\pi \in \mathcal{B}_{n+1,n+1}} \pi.
\]

Alternatively (R. & Westbury):

\[
\{ \text{ev}_n(\pi) \mid \pi \text{ is } (n+1)\text{-noncrossing} \}
\]

is a basis of \( \text{Hom}_{\text{Sp}_{2n}} ( V \otimes r, V \otimes s ) \).
$(n + 1)$-noncrossing perfect matchings

$\mathcal{B}_{r,s}$ is the set of perfect matchings of $1, \ldots, r + s$.

\[
\begin{align*}
 r &= 3 \\
 s &= 5
\end{align*}
\]
(n + 1)-noncrossing perfect matchings

\( \mathcal{B}_{r,s} \) is the set of perfect matchings of 1, \ldots, \( r + s \).

\( r = 3 \)

\( s = 5 \)

A perfect matching is \((n + 1)\)-noncrossing, if there is no set of \( n + 1 \) mutually crossing arcs.
$(n + 1)$-noncrossing perfect matchings

$\mathcal{B}_{r,s}$ is the set of perfect matchings of $1, \ldots, r + s$.

$r = 3$

$s = 5$

3-noncrossing

A perfect matching is $(n + 1)$-noncrossing, if there is no set of $n + 1$ mutually crossing arcs.

a 3-crossing
**Sp\(_{2n}\), natural representation (Brauer, Weyl)**

Let \(B_{r,s}\) be the set of perfect matchings of 1, \ldots , \(r + s\).

**FFT:** \(\text{Hom}_{Sp_{2n}} (V \otimes r, V \otimes s) = 0\) when \(r + s\) odd, and the evaluation functor

\[ ev_n : B_{r,s} \rightarrow \text{Hom}_{Sp_{2n}} (V \otimes r, V \otimes s) \]

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\[ E(n + 1) = \sum_{\pi \in B_{n+1,n+1}} \pi. \]

Alternatively (R. & Westbury):

\[ \{ ev_n(\pi) \mid \pi \text{ is } (n + 1)-\text{noncrossing} \} \]

is a basis of \(\text{Hom}_{Sp_{2n}} (V \otimes r, V \otimes s)\).
Sketch of proof of the second fundamental theorem

\[ \text{ev}_n \left( E(n + 1) \right) = \text{ev}_n \left( \sum_{\pi \in \mathcal{B}_{n+1,n+1}} \pi \right) = 0 \]

(not completely trivial)
Sketch of proof of the second fundamental theorem

- \( \text{ev}_n (E(n + 1)) = \text{ev}_n \left( \sum_{\pi \in \mathcal{B}_{n+1,n+1}} \pi \right) = 0 \)
  
  (not completely trivial)

example: \( n = 1 \):

\[
\text{ev}_n \left( \begin{array}{c}
\square \\
\square \\
\square
\end{array} + \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array} + \begin{array}{c}
\square \\
\square
\square
\end{array} \right)
\]

\[
= u \otimes v \mapsto u \otimes v - v \otimes u + (b_1 \otimes b_1^* + b_2 \otimes b_2^*) \langle u, v \rangle.
\]
Sketch of proof of the second fundamental theorem

- $\text{ev}_n \left( E(n+1) \right) = \text{ev}_n \left( \sum_{\pi \in \mathcal{B}_{n+1,n+1}} \pi \right) = 0$

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- $\dim \mathcal{C} \mathcal{B}_{r,s}/\langle E(n+1) \rangle \geq \dim \text{Hom}_{\text{Sp}_{2n}}(V^\otimes r, V^\otimes s)$

  (since $\text{ev}_n$ is surjective and $\langle E(n+1) \rangle \subseteq \ker \text{ev}_n$)
Sketch of proof of the second fundamental theorem

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  \( s = 0 \) suffices!
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  \( s = 0 \) suffices!

\[
\dim \text{Hom}_{\text{Sp}_{2n}} \left( V \otimes r, \mathbb{C} \right) = \#(n + 1)\text{-noncrossing perfect matchings of } 1, \ldots, r \quad \text{(Sundaram)}
\]
Sketch of proof of the second fundamental theorem

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\dim \text{Hom}_{\text{Sp}_{2n}} (V \otimes r, \mathbb{C}) \quad \text{(Sundaram)}
\]
\[
= \#(n + 1)\text{-noncrossing perfect matchings of } 1, \ldots, r
\]

\[
\dim \mathbb{C}B_{r,s} / \langle E(n + 1) \rangle
\]
\[
\leq \#(n + 1)\text{-noncrossing perfect matchings of } 1, \ldots, r
\]

(exactly one summand of $E(n + 1)$ is $(n + 1)$-noncrossing)
$\text{Sp}_{2n}$, symmetric powers of natural representation
(Rubey & Westbury)

Let $\mathcal{B}_{r,s}^k$ be the set of perfect matchings of $1, \ldots, k \cdot (r + s)$, such that

- points in a block are not matched and
- arcs originating from one block do not cross.

$r + s = 4, \ k = 2$
$Sp_{2n}$, symmetric powers of natural representation (Rubey & Westbury)

Let $\mathcal{B}_{r,s}^k$ be the set of perfect matchings of $1, \ldots, k \cdot (r + s)$, such that

- points in a block are not matched and
- arcs originating from one block do not cross.

$r + s = 4, \ k = 2$

**SFT:** Let $W = S^k(V) = V^{\otimes k}/(u \otimes v - v \otimes u)$ and $ev_n : \mathbb{C}\mathcal{B}_{r,s}^k \to Hom_{Sp_{2n}}(W^{\otimes r}, W^{\otimes s})$. Then

$$\{ ev_n(\pi) \mid \pi \text{ is } (n + 1)-\text{noncrossing} \}$$

is a basis of $Hom_{Sp_{2n}}(W^{\otimes r}, W^{\otimes s})$. 
The cyclic sieving phenomenon
Let $X$ be a finite set and $\langle c \rangle$ a cyclic group with $r$ elements, acting on $X$. ('rotation')
The cyclic sieving phenomenon

- Let \( X \) be a finite set and \( \langle c \rangle \) a cyclic group with \( r \) elements, acting on \( X \). (‘rotation’)
- Let \( P(q) \) a polynomial with non-negative integer coefficients, such that for any primitive \( r \)th root of unity \( \zeta \)

\[
P(\zeta^d) = \# \text{fixed points of } c^d.
\]
The cyclic sieving phenomenon

Let $X$ be a finite set and $\langle c \rangle$ a cyclic group with $r$ elements, acting on $X$. (‘rotation’)

Let $P(q)$ a polynomial with non-negative integer coefficients, such that for any primitive $r$th root of unity $\zeta$

$$P(\zeta^d) = \#\text{fixed points of } c^d.$$ 

Then $(X, \langle c \rangle, P(q))$ exhibits the ‘cyclic sieving phenomenon’

(Reiner, Stanton & White)
Example: noncrossing perfect matchings

Let $X$ be the set of noncrossing perfect matchings of $1, \ldots, 2r$: 

\[
|X| = \frac{1}{r+1} \binom{2r}{r} \quad (\text{Catalan})
\]

\[
P(q) = \frac{1}{r+1} q^{\binom{2r}{r}}
\]

in case one cannot guess $P(q)$, it is usually hard to find. 

\[
m q = 1 + q + \cdots + q^{m-1} 
= \frac{1 - q^m}{1 - q} 
= [m] q + [m] q^{-1} + \cdots + [m] q^{m-1} 
\]
Example: noncrossing perfect matchings

- Let $X$ be the set of noncrossing perfect matchings of $1, \ldots, 2r$:

- $|X| = \frac{1}{r+1} {2r \choose r}$  

  $1, 2, 5, 14, 42, \ldots$ \hspace{1cm} (Catalan)
Example: noncrossing perfect matchings

- Let $X$ be the set of noncrossing perfect matchings of $1, \ldots, 2r$:

- $|X| = \frac{1}{r+1} \binom{2r}{r}$

- $P(q) = \frac{1}{[r+1]_q} \left[ \begin{array}{c} 2r \\ r \end{array} \right]_q$ (Reiner, Stanton & White)

- $1, 1 + q^2, 1 + q^2 + q^3 + q^4 + q^6, \ldots$

- $1, 2, 5, 14, 42, \ldots$ (Catalan)

- $[m]_q = 1 + q + \ldots + q^{m-1}$

- $[m]_q! = [m]_q \ldots [2]_q [1]_q$

- $\left[ \begin{array}{c} m \\ k \end{array} \right]_q = \frac{[m]_q!}{[k]_q! [m-k]_q!}$
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$(\text{Reiner, Stanton \& White})$

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- $(\text{Reiner, Stanton & White})$

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a miracle!

in case one cannot guess $P(q)$, it is usually hard to find...

$$[m]_q = 1 + q + \ldots q^{m-1}$$

$$[m]_q! = [m]_q \cdot [2]_q [1]_q$$

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The cyclic sieving phenomenon

**Theorem:** Let $X \subset U$ be a basis of the module $\rho: \mathfrak{S}_r \to \text{End}(U)$, which is permuted by the long cycle $c = (1, 2, \ldots, r)$. Then

$$(X, \langle c \rangle, \text{fd ch}(\rho))$$

exhibits the cyclic sieving phenomenon.
The cyclic sieving phenomenon

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$$(X, \langle c \rangle, \text{fd ch}(\rho))$$

exhibits the cyclic sieving phenomenon.

- the Frobenius character of $\rho$ is

$$\text{ch}(\rho) = \frac{1}{r!} \sum_{\pi \in \mathfrak{S}_r} \text{tr} \rho(\pi) p_{\lambda(\pi)}$$

$\text{tr}$ the trace,
$\lambda(\pi) = (\lambda_1, \lambda_2, \ldots)$ the cycle type of $\pi$,
$p_k = x_1^k + x_2^k + \ldots$, $p_\lambda = p_{\lambda_1} \cdot p_{\lambda_2} \cdots$
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$\text{tr}$ the trace,
$\lambda(\pi) = (\lambda_1, \lambda_2, \ldots)$ the cycle type of $\pi$,
$p_k = x_1^k + x_2^k + \ldots$, $p_\lambda = p_{\lambda_1} \cdot p_{\lambda_2} \cdots$

$\triangleright$ the ‘fake degree’ polynomial $\text{fd}$ is

$$\text{fd}(s_\lambda) = \sum_{T \text{ a standard Young tableau of shape } \lambda} q^{\text{maj } T}$$
The cyclic sieving phenomenon

Theorem: (Sundaram), (Tokuyama)

\[ \text{ch} \, \text{Hom}_{\text{Sp}(2n)}(0, 2r) = \sum_{\lambda \vdash 2r, \ell(\lambda) \leq 2n} S_{\lambda^t} \]
Theorem: (Sundaram), (Tokuyama)

\[ \text{ch} \, \text{Hom}_{\text{Sp}(2n)}(0, 2r) = \sum_{\substack{\lambda \vdash 2r \\ \text{columns of even length} \\ \ell(\lambda) \leq 2n}} s_{\lambda^t} \]

Corollary: Let \( X \) be the set of \((n + 1)\)-noncrossing perfect matchings of \(1, \ldots, 2r\) and let \( c \) be rotation by one element. Then

\[ (X, \langle c \rangle, \text{fd ch} \, \text{Hom}_{\text{Sp}(2n)}(0, 2r)) \]

exhibits the cyclic sieving phenomenon.
Further results

- natural representation of $\mathfrak{S}_n$
  (a permutation matrix $g$ acts on $v \in \mathbb{C}^n$ as $g \cdot v$)

The morphisms of the diagram category are set partitions.
The set of set partitions into at most $n$ blocks is a basis.

(Halverson, Martin, Ram)
Further results

- natural representation of $\mathfrak{S}_n$
  (a permutation matrix $g$ acts on $v \in \mathbb{C}^n$ as $g \cdot v$)
  The morphisms of the diagram category are set partitions.
  The set of set partitions into at most $n$ blocks is a basis.
  (Halverson, Martin, Ram)

- adjoint representation of $\mathfrak{GL}_n$
  (a matrix $g$ acts on $v \in \text{Mat}_{n,n}(\mathbb{C})$ as $g \cdot v \cdot g^{-1}$)
  The morphisms of the diagram category are permutations (or directed matchings).
  The set of permutations with length of longest decreasing subsequence at most $n$ is a basis.
  basis invariant under rotation still unknown.
  (Rubey & Westbury)
Promotion
\( n \)-symplectic oscillating tableaux

An \( n \)-symplectic oscillating tableau of shape \( \mu \) is a sequence of partitions

\[
(\emptyset = \mu_0, \mu_1, \ldots, \mu_r = \mu)
\]

such that consecutive partitions differ by precisely one cell and each partition has at most \( n \) parts.
n-symplectic oscillating tableaux

...are the highest weight words for the representation of $\text{Sp}_{2n}$ on $V^\otimes r$.

Combinatorics ... are (for $\mu = \emptyset$) in bijection (Sundaram) with $(n + 1)$-noncrossing perfect matchings of $1, \ldots, r$. 
Promotion

The promotion of a highest weight word $w = w_1 \ldots w_r$ of $V^\otimes r$ can be obtained as follows:

- let $w'$ be $w$ without its first letter,
- let $w''$ be the unique highest weight word in the same component as $w'$
- obtain $\text{pr } w$ by appending the unique letter to $w''$ such that $\text{pr } w$ and $w$ have the same weight.
Promotion

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- obtain $\text{pr}_r w$ by appending the unique letter to $w''$ such that $\text{pr}_r w$ and $w$ have the same weight.

\text{a miracle:}
when $w$ has weight 0, $\text{Sun pr}_r w = \text{rot Sun}_r w$
Sketch of proof of the miracle

- Interpret promotion as a generator \((s_1, rs_2, r)\) of Henriques and Kamnitzer’s cactus group.
- ‘local rules’ for \(w \mapsto \text{pr} w\) are known. \((\text{van Leeuven, Lenart})\)
- Determine ‘local rules’ for the map \(\text{Sun}^{-1} M \mapsto \text{Sun}^{-1} \text{rot} M\).
- Show that these coincide.
Promotion via local rules (van Leeuven)

\[
\begin{array}{cccccccccc}
0 & 1 & 11 & 21 & 21 & 11 & 21 & 11 & 1 & 0 \\
0 & 1 & 2 & 21 & 22 & 21 & 31 & 21 & 2 & 1 & 0 \\
\end{array}
\]

\[\lambda \]
\[\nu\]

\[\kappa \]
\[\mu\]

\[\mu = \text{dom}_n(\kappa + \nu - \lambda)\]

\[\lambda = \text{dom}_n(\kappa + \nu - \mu)\]
Sundaram’s bijection

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\[ c = \begin{array}{c}
\lambda \\
\downarrow \kappa \\
\downarrow \\
\mu
\end{array} \quad \text{or} \quad \begin{array}{c}
\lambda \\
\downarrow \\
\downarrow \mu
\end{array} \]

\[ \mu' = \text{dom}_{\mathfrak{S}_n}(\kappa' + \nu' - \lambda') \]

\[ \lambda' = \text{dom}_{\mathfrak{S}_n}(\kappa' + \nu' - \mu') \]

\[ \mu = \lambda + \epsilon_1 \]

\[ \lambda = \mu - \epsilon_1 \]