On the combinatorics of the symplectic group

Martin Rubey
(joint with Bruce Westbury and with
Stephan Pfannerer and Bruce Westbury)

26.5.2017

Invariants

Background: Invariants

- G a group of matrices - GL_n, O_n, Sp_{2n}, $\mathfrak{S}_n$
- V a G-module - natural or adjoint representation

Goal: describe the ring of polynomial invariants

$$\mathbb{C}[V]^G = \{f \text{ polynomial in the coordinates of } V : \\ \forall g \in G, v \in V : f(gv) = f(v)\}.$$

First Fundamental Theorem (FFT): explicit generating system

Second Fundamental Theorem (SFT): relations between generators

Background: Invariants

Equivalently (in characteristic 0), describe

$$\mathrm{Hom}_G\left(V^{\otimes r}, V^{\otimes s}\right) = \{f \in \mathrm{Hom}\left(V^{\otimes r}, V^{\otimes s}\right) : \\ \forall g \in G, \mathbf{v} \in V^{\otimes r} : f(g\mathbf{v}) = gf(\mathbf{v})\}.$$

FFT: explicit generating system

SFT: linear relations between generators

Background: Ancestors

David Hilbert, 'Über die Theorie der algebraischen Formen', 1890.

14th problem: is the ring of invariants finitely generated?

Hermann Weyl, 'The classical groups - their invariants and representations', 1939.

first and second fundamental theorem for the classical groups and their natural representation

Richard Brauer, 'On algebras which are connected with the semisimple continuous groups', 1937.

combinatorial description of the invariants of the orthogonal and the symplectic group

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Emmy Noether, 'Der Endlichkeitssatz der Invarianten endlicher Gruppen', 1916.
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Emmy Noether, 'Der Endlichkeitssatz der Invarianten endlicher Gruppen', 1916.
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Judith Braunsteiner, 'A Sundaram bijection for the odd orthogonal groups', 2017?.

combinatorial description of the invariants of the odd orthogonal group in low dimension

Sheila Sundaram, 'On the combinatorics of representations of the symplectic group', 1986.

combinatorial description of the invariants of the symplectic group in low dimension

Background: classical results

- natural representation of GL_n (Schur, 1901)
  (a matrix g acts on $v \in \mathbb{C}^n$ as $g \cdot v$)
- natural representation of O_n, Sp_{2n} (Weyl, 1924)
- natural representation of $\mathfrak{S}_n$
- adjoint representation of GL_n
  (a matrix g acts on $v \in Mat_{n,n}(\mathbb{C})$ as $g \cdot v \cdot g^{-1}$)

Background: GL_n, natural representation (Schur, Weyl)

FFT: $\mathrm{Hom}_{\mathrm{GL}_n}(V^{\otimes r}, V^{\otimes s}) = 0$ for $r \neq s$,
and the algebra homomorphism

$$\mathrm{ev}_n : \mathbb{C}\mathfrak{S}_r \to \mathrm{Hom}_{\mathrm{GL}_n}\left(V^{\otimes r}, V^{\otimes r}\right)$$
$$\mathrm{ev}_n(\sigma) = v_1 \otimes \cdots \otimes v_r \mapsto v_{\sigma^{-1}1} \otimes \cdots \otimes v_{\sigma^{-1}r}$$

is surjective.

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$$E(n+1) = \sum_{\pi \in \mathfrak{S}_{n+1}} \varepsilon(\pi)\pi.$$

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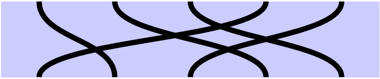
Alternatively:

$$\{\mathsf{ev}_n(\pi) \mid \text{longest decreasing subsequence in } \pi \text{ has length at most } n\}$$

is a basis of $\mathrm{Hom}_{\mathrm{GL}_n}(V^{\otimes r}, V^{\otimes r})$.

Idea: algebraic combinatorics

algebra		combinatorics
$\mathrm{Hom}_{\mathrm{GL}_n}(V^{\otimes r}, V^{\otimes r})$	$\longleftrightarrow$	 $(r = 5,\ n \geq 2)$
composition	$\longleftrightarrow$	stack
tensor product	$\longleftrightarrow$	draw side by side

(modulo $\ker \mathrm{ev}_n$)

Sp_{2n}, natural representation (Brauer, Weyl)

Let $\mathfrak{B}_{r,s}$ be the set of perfect matchings of $1, \dots, r+s$.

FFT: $\mathrm{Hom}_{\mathrm{Sp}_{2n}}(V^{\otimes r}, V^{\otimes s}) = 0$ when $r+s$ odd, and the evaluation functor

$$\mathrm{ev}_n : \mathfrak{B}_{r,s} \to \mathrm{Hom}_{\mathrm{Sp}_{2n}}\left(V^{\otimes r}, V^{\otimes s}\right)$$

is full (i.e., surjective on objects).

SFT: The kernel of ev_n is generated by

$$E(n+1) = \sum_{\pi \in \mathfrak{B}_{n+1,n+1}} \pi.$$

Alternatively (R. & Westbury):

$$\{\mathrm{ev}_n(\pi) \mid \pi \text{ is } (n+1)\text{-noncrossing}\}$$

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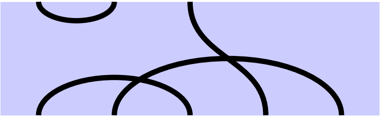
Brauer's category

$\mathfrak{B}_{r,s}$ is the set of perfect matchings of $1, \ldots, r+s$.



$r = 3$

$s = 5$

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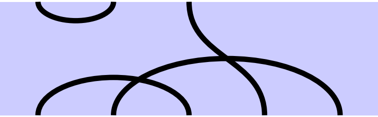

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$\mathbb{C}\mathfrak{B}_{r,s}$ is the set of morphisms of 'Brauer's category':

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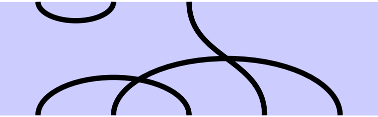

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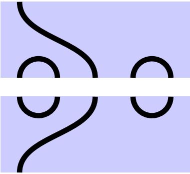
When stacking, loops may occur. Remove them, then multiply the result with $\delta = -2n$ for each loop:

 $= \delta^2$

The evaluation functor

- $b_1, \ldots, b_{2n}$ a basis of V,
- $\langle\ ,\ \rangle$ a skew symmetric bilinear form,
- $b_1^*, \ldots, b_{2n}^*$ the dual basis, regarded as basis of V: $\langle b_i^*, b_j \rangle = \delta_{i,j}$.

Define

$$\mathrm{ev}_n : \mathbb{C}\mathfrak{B}_{r,s} \to \mathrm{Hom}_{\mathrm{Sp}_{2n}}\left(V^{\otimes r}, V^{\otimes s}\right)$$

by

$\mathrm{ev}_n\Big($  $\Big) = u \otimes v \mapsto -v \otimes u$

$\mathrm{ev}_n\Big($  $\Big) = 1 \mapsto \sum_i b_i \otimes b_i^*$

$\mathrm{ev}_n\Big($  $\Big) = u \otimes v \mapsto \langle u, v \rangle.$

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(exercise: these are Sp_{2n}-invariants.)

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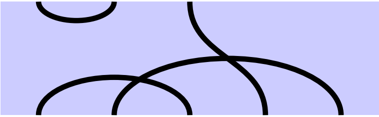

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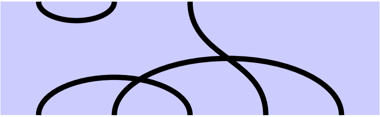

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A perfect matching is $(n+1)$-noncrossing,
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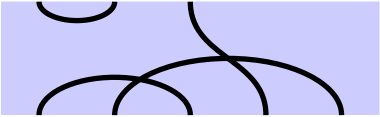

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3-noncrossing

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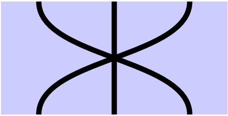

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Sketch of proof of the second fundamental theorem

- $\mathrm{ev}_n\left(E(n+1)\right) = \mathrm{ev}_n\left(\sum_{\pi \in \mathfrak{B}_{n+1,n+1}} \pi\right) = 0$

  (not completely trivial)

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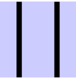
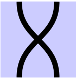
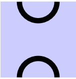
  example: $n = 1$:

  $\mathrm{ev}_n\Big($  $+$  $+$  $\Big)$

  $= u \otimes v \mapsto u \otimes v - v \otimes u + (b_1 \otimes b_1^* + b_2 \otimes b_2^*)\langle u, v\rangle.$

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$$\dim \mathrm{Hom}_{\mathrm{Sp}_{2n}}\left(V^{\otimes r}, \mathbb{C}\right) = \#(n+1)\text{-noncrossing perfect matchings of } 1, \ldots, r \qquad \text{(Sundaram)}$$

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  $$\begin{aligned} &\dim \mathrm{Hom}_{\mathrm{Sp}_{2n}}\left(V^{\otimes r}, \mathbb{C}\right) \qquad \text{(Sundaram)} \\ &\quad = \#(n+1)\text{-noncrossing perfect matchings of } 1, \ldots, r \end{aligned}$$

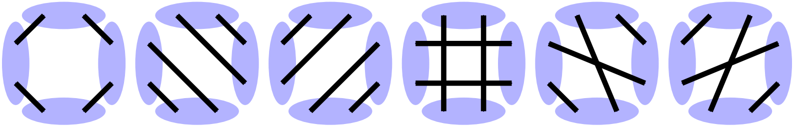
  $$\begin{aligned} &\dim \mathbb{C}\mathfrak{B}_{r,s}/\langle E(n+1)\rangle \\ &\quad \leq \#(n+1)\text{-noncrossing perfect matchings of } 1, \ldots, r \end{aligned}$$

  (exactly one summand of $E(n+1)$ is $(n+1)$-noncrossing)

Sp_{2n}, symmetric powers of natural representation (Rubey & Westbury)

Let $\mathfrak{B}^k_{r,s}$ be the set of perfect matchings of $1, \dots, k \cdot (r+s)$, such that

- points in a block are not matched and
- arcs originating from one block do not cross.



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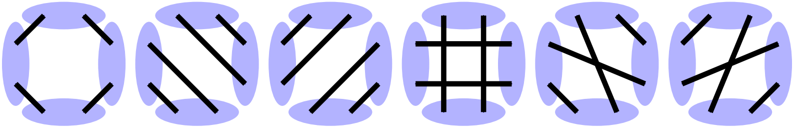

$r + s = 4$, $k = 2$

SFT: Let $W = S^k(V) = V^{\otimes k}/(u \otimes v - v \otimes u)$ and $\mathrm{ev}_n : \mathbb{C}\mathfrak{B}^k_{r,s} \to \mathrm{Hom}_{\mathrm{Sp}_{2n}}(W^{\otimes r}, W^{\otimes s})$. Then

$$\{\mathrm{ev}_n(\pi) \mid \pi \text{ is } (n+1)\text{-noncrossing}\}$$

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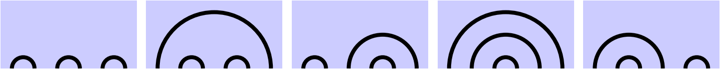
Then $\big(X, \langle c \rangle, P(q)\big)$ exhibits the 'cyclic sieving phenomenon'

(Reiner, Stanton & White)

Example: noncrossing perfect matchings

- Let X be the set of noncrossing perfect matchings of $1, \ldots, 2r$:



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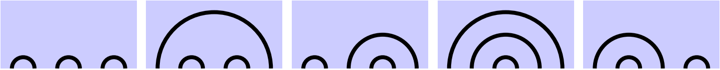

- $|X| = \frac{1}{r+1}\binom{2r}{r}$ $1, 2, 5, 14, 42, \ldots$ (Catalan)

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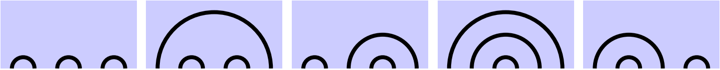

- $|X| = \frac{1}{r+1}\binom{2r}{r}$ $\quad 1, 2, 5, 14, 42, \ldots$ (Catalan)
- $P(q) = \frac{1}{[r+1]_q}\begin{bmatrix} 2r \\ r \end{bmatrix}_q$ (Reiner, Stanton & White)

$1, 1+q^2, 1+q^2+q^3+q^4+q^6, \ldots$

$$[m]_q = 1 + q + \ldots q^{m-1}$$

$$[m]_q! = [m]_q \ldots [2]_q[1]_q$$

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{[m]_q!}{[k]_q![m-k]_q!}$$

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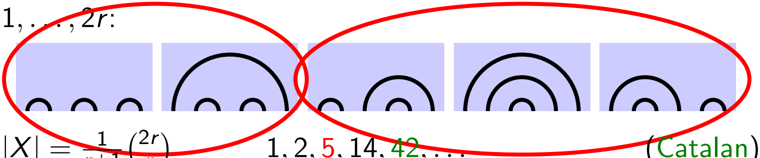

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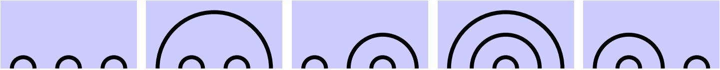

- $|X| = \frac{1}{r+1}\binom{2r}{r}$ $\quad 1, 2, 5, 14, 42, \ldots$ (Catalan)
- $P(q) = \frac{1}{[r+1]_q}\begin{bmatrix}2r\\r\end{bmatrix}_q$ (Reiner, Stanton & White)

$1, 1+q^2, 1+q^2+q^3+q^4+q^6, \ldots$

a miracle!

in case one cannot guess $P(q)$, it is usually hard to find...

$$[m]_q = 1 + q + \ldots q^{m-1}$$

$$[m]_q! = [m]_q \ldots [2]_q[1]_q$$

$$\begin{bmatrix}m\\k\end{bmatrix}_q = \frac{[m]_q!}{[k]_q![m-k]_q!}$$

The cyclic sieving phenomenon

Theorem: Let $X \subset U$ be a basis of the module $\rho \colon \mathfrak{S}_r \to \mathrm{End}(U)$, which is permuted by the long cycle $c = (1, 2, \dots, r)$. Then

$$\big(X, \langle c\rangle, \mathbf{fd\,ch}(\rho)\big)$$

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- the Frobenius character of ρ is

$$\mathbf{ch}(\rho) = \frac{1}{r!} \sum_{\pi \in \mathfrak{S}_r} \mathbf{tr}\,\rho(\pi) p_{\lambda(\pi)}$$

tr the trace,
$\lambda(\pi) = (\lambda_1, \lambda_2, \ldots)$ the cycle type of π,
$p_k = x_1^k + x_2^k + \ldots$, $p_\lambda = p_{\lambda_1} \cdot p_{\lambda_2} \cdots$

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- the 'fake degree' polynomial **fd** is

$$\mathbf{fd}(s_\lambda) = \sum_{T \text{ a standard Young tableau of shape } \lambda} q^{\operatorname{maj} T}$$

The cyclic sieving phenomenon

Theorem: (Sundaram), (Tokuyama)

$$\mathbf{ch}\,\mathrm{Hom}_{\mathrm{Sp}(2n)}(0, 2r) = \sum_{\substack{\lambda \vdash 2r \\ \text{columns of even length} \\ \ell(\lambda) \leq 2n}} s_{\lambda^t}$$

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Corollary: Let X be the set of $(n+1)$-noncrossing perfect matchings of $1, \ldots, 2r$ and let c be rotation by one element. Then

$$\left(X, \langle c \rangle, \mathbf{fd}\,\mathbf{ch}\,\mathrm{Hom}_{\mathrm{Sp}(2n)}(0, 2r)\right)$$

exhibits the cyclic sieving phenomenon.

Further results

- natural representation of $\mathfrak{S}_n$
  (a permutation matrix g acts on $v \in \mathbb{C}^n$ as $g \cdot v$)
  The morphisms of the diagram category are set partitions.
  The set of set partitions into at most n blocks is a basis.
  (Halverson, Martin, Ram)

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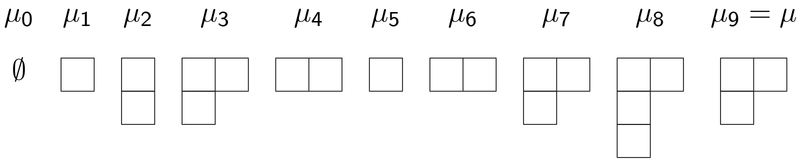
  (Halverson, Martin, Ram)
- adjoint representation of GL_n
  (a matrix g acts on $v \in Mat_{n,n}(\mathbb{C})$ as $g \cdot v \cdot g^{-1}$)
  The morphisms of the diagram category are permutations (or directed matchings).
  The set of permutations with length of longest decreasing subsequence at most n is a basis.
  basis invariant under rotation still unknown.
  (Rubey & Westbury)

Promotion

n-symplectic oscillating tableaux



An n-symplectic oscillating tableau of shape μ is a sequence of partitions

$$(\emptyset = \mu_0, \mu_1, \ldots, \mu_r = \mu)$$

such that consecutive partitions differ by precisely one cell and each partition has at most n parts.

n-symplectic oscillating tableaux

algebra ... are the highest weight words for the representation of Sp_{2n} on $V^{\otimes r}$.

combinatorics ... are (for $\mu = \emptyset$) in bijection (Sundaram) with $(n+1)$-noncrossing perfect matchings of $1, \ldots, r$.

Promotion

The promotion of a highest weight word $w = w_1 \ldots w_r$ of $V^{\otimes r}$ can be obtained as follows:

- let w' be w without its first letter,
- let w'' be the unique highest weight word in the same component as w'
- obtain $\operatorname{pr} w$ by appending the unique letter to w'' such that $\operatorname{pr} w$ and w have the same weight.

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a miracle:

when w has weight 0, $\operatorname{Sun} \operatorname{pr} w = \operatorname{rot} \operatorname{Sun} w$

Sketch of proof of the miracle

- Interpret promotion as a generator $(s_{1,r}s_{2,r})$ of Henriques and Kamnitzer's cactus group.
- 'local rules' for $w \mapsto \operatorname{pr} w$ are known. (van Leeuven, Lenart)
- Determine 'local rules' for the map $\operatorname{Sun}^{-1} M \mapsto \operatorname{Sun}^{-1} \operatorname{rot} M$.
- Show that these coincide.

Promotion via local rules (van Leeuven)

$$\begin{array}{ccccccccccc} 0 & 1 & 11 & 21 & 2 & 21 & 11 & 21 & 11 & 1 & 0 \\ & 0 & 1 & 2 & 21 & 22 & 21 & 31 & 21 & 2 & 1 & 0 \end{array}$$

$$\begin{array}{cc} \lambda & \nu \\ \kappa & \mu \end{array}$$

$$\mu = \mathrm{dom}_{\mathfrak{H}_n}(\kappa + \nu - \lambda)$$

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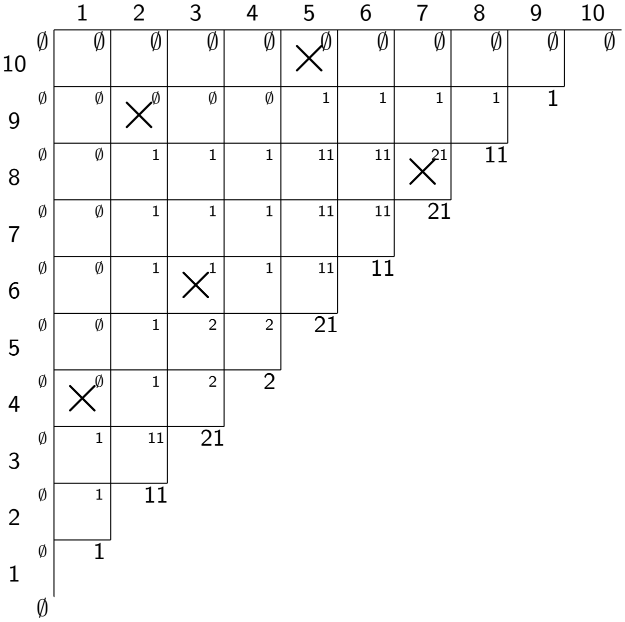
Sundaram's bijection



$$
c = \begin{tikzcd} \lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\ \kappa \arrow[r] & \mu \end{tikzcd} \quad \text{or} \qquad c = \begin{tikzcd} \lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxtimes"] & \lambda \arrow[d] \\ \lambda \arrow[r] & \mu \end{tikzcd}
$$

$\mu' = \mathrm{dom}_{\mathfrak{S}_n}(\kappa' + \nu' - \lambda')$	$\mu = \lambda + \epsilon_1$
$\lambda' = \mathrm{dom}_{\mathfrak{S}_n}(\kappa' + \nu' - \mu')$	$\lambda = \mu - \epsilon_1$