Van der Waerden Variations

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(joint work with João Guerreiro e Imre Ruzsa)

Coloring a complete graph

Theorem (Ramsey - 1930)

Given positive integers r and k if we r-color the edges of the complete graph with at least $N = N(r, k)$ vertices, we can always find monochromatic complete subgraphs of order k.

This result implies that we can always find a monochromatic copy of any finite subgraph H.

Van der Waerden and arithmetic progression

The next statement is, with Ramsey´s theorem for graphs, one of the most influential results in the combinatorics of unavoidable structures.

Theorem (van der Waerden - 1927)

Any finite partition of the positive integers contains large monochromatic arithmetic progressions.

We could not ask for an infinite monochromatic progressions, as a coloring such as 01100011110000 shows.

Density version - no coloring involved

Theorem (Szemerédi - 1969)

Let $A \subset \mathbb{N}$ with positive upper density, i.e.,

$$d^*(A) := \limsup_{N \to \infty} \frac{\#\{n \in A : n \leq N\}}{N} > 0.$$

Then for any integer $k \geq 1$, the set A contains an arithmetic progression $a, a+r, a+2r, \ldots, a+(k-1)r$ of length k, where a, r are positive integers.

This result says that any "positive fraction" of the positive integers will always contain arbitrarily long arithmetic progressions $a, a+r, a+2r, \ldots, a+(k-1)r$

Does density implies structure?

We should note that the natural density version for Ramsey's theorem on graphs fails.

In fact, the complete bipartite graph $K(n, n)$ contains a positive fraction of all possible edges: $|E(K)| = n^2$. But, $K(n, n)$ contains no triangle.

Unlike what happens for graphs, arithmetic progressions admit density versions.

Large sets

A set $L \subseteq \mathbb{N}$ is d-large if for any d-coloring $\alpha : \mathbb{N} \to [1, d]$ there are arbitrarily large monochromatic AP's with common difference $r \in L$. The set of odd numbers is not large as the coloring 010101... shows.

A necessary condition for a set L to be large is: given any positive integer n, L contains a multiple of n.

Theorem (Brown,Graham, Landman)

If $A = A_1 \cup \cdots A_n$ and A is large, then some A_i is large.

Problem (open)

Every 2-large set is necessarily k-large for every $k > 2$?

Acessible sets - a weaker notion

A set $S \subset \mathbb{N}$ is accessible if for any finite coloring $\alpha : \mathbb{N} \to [1, d]$ there are arbitrarily large monochromatic sets $A = \{a_1, a_2, \ldots, a_n\}$ such that $a_{i+1} - a_i \in S$. We will say that A is a monochromatic S-sequence. Two natural ways to construct accessible sets are:

- An infinite difference set $S = A - A$ with $A \subset \mathbb{N}$ infinite.
- Use the fact that every "polynomial set" is large (Bergelson, Leibman). The set of the perfect squares or the cubes are large sets.

Chromatic intersective

A set $S \subset \mathbb{N}$ is chromatic intersective if for any finite partition of

$$\mathbb{N} = A_1 \cup \cdots \cup A_d,$$

there is $1 \leq i \leq d$ such that $(A_i - A_i) \cap S \neq \emptyset$.

It is clear that any accessible set is also chromatic intersective.

Theorem

Every chromatic intersective set $S \subset \mathbb{N}$ is also accessible.

Proof Let $k \geq 2$ be the maximal length for some monochromatic S-sequence over all possible finite colorings. Consider a coloring $\alpha : \mathbb{N} \to [1, r]$ where this length can not be improved. Re-color now the last elements in all maximal S-sequences.

An ultrafilter property

Theorem
If $L = L_1 \cup L_2$ is accessible, then L_1 is accessible or L_2 is accessible.

Proof Suppose L_1 and L_2 are not accessible. By the previous theorem, both sets are not chromatic intersective. Let $\alpha : \mathbb{N} \to [1, d]$ avoids differences from L_1 and $\beta : \mathbb{N} \to [1, r]$ avoids differences from L_2. Define $\gamma(n) = (\alpha(n), \beta(n))$ avoids differences from $L = L_1 \cup L_2$. This implies that L is not chromatic intersective which contradicts the hypothesis.

Density intersective sets

A set $S \subset \mathbb{N}$ is *density intersective* if, given any set $A \subset \mathbb{N}$ with positive (upper) density, we have $(A - A) \cap S \neq \emptyset$. S intersects the difference $A - A$ of all positive density sets A.

Theorem

Let $S \subset \mathbb{N}$ be a density intersective set and $A \subset \mathbb{N}$ a positive density set. Then, the set A contains arbitrarily large S-sequences.

Proof Choose the set of positive density which gives the smallest possible S-sequences.
We define $ord(n)$ to be the maximal length over all S-sequences $x_0 = n, x_1, \cdots, x_k$, $x_i \in A$.
Let $A_k = \{n \in \mathbb{N} : ord(n) = k\}$ be the set of elements of $A_0 = \{n \in \mathbb{N} : ord(n) = 0\}$, which is not empty!

$$A = A_k \cup A_0 \cup B.$$

Infinite distance graph

Given a set of positive integers D we define the *distance graph* G_D where $V(G) = \mathbb{Z}$ and $E(G) = \{(x, y) \in V \times V : |x - y| \in D\}$.

If $D \subset \mathbb{N}$ is 2-intersective, then the associate distance graph G_D has infinite chromatic number.

If we take $D = \{n^3 : n \geq 1\}$ then G_D has infinite chromatic number but (by Fermat Last theorem) contains no triangle.

Large monochromatic paths

What can we say about the structure of this graph?

Theorem

Let $D \subset \mathbb{N}$ such that G_D as infinite chromatic number and a finite d-coloring of the vertices $\alpha : V \to [1, d]$. There exist arbitrarily large monochromatic upwards paths.

In fact, the result is true for any graph with infinite chromatic number.

An elementary construction

Theorem

If $S = \left\{m^{2k-1} : k \in \mathbb{N}\right\}$, for $m \geq 5$, then $S - S$ is accessible and is not 3-large.

Remark: Jungic proved an analogous result using a non explicit construction from Furstenberg.

Proof $S - S$ is accessible by the previous lemma. We will show that $S - S$ is not 3-large.
Let $f_i(n)$ be the i-th digit in the base m representation of

$$n = \sum f_i(n) m^i.$$

An elementary construction

Consider the following 3-coloring of the positive integers,

$$f(n) = |\{i : f_{2i}(n) = 2\}| \bmod 3.$$

There are no monochromatic AP with length $n > m^2 + m + 1$. We conclude that $S - S$ is not 3-large.

Note: We can not decide if this sets are 2-large.

Infinite monochromatic walks

If we take $S = A - A$ with $A \subset N$ an infinite set, then given any finite coloring there will always exist infinite monochromatic S-walks.

Given a set $S \subset \mathbb{N}$ we define its *order* $ord(S)$ to be the larger positive integer d for which every d-coloring contains infinite monochromatic S-walks. If no such larger d exists, then we say that $ord(S) = \infty$ and that $S \subset \mathbb{N}$ is a *infinite ladder*.

Theorem

If $S = \{n^2 : n \in \mathbb{N}\}$ is the set of squares, then $ord(S) \geq 2$.

The proof uses the pythagorean triples description.

Infinite monochromatic walks

The next result says that the gaps in an infinite ladder can not increase to much.

Theorem

Let $S = \{s_1, s_2, \cdots\}$ be an infinite ladder. Then, $\liminf(s_{i+k} - s_i)$ *is finite.*

Proof $[k = 1]$ We will use 4 colors. We start by coloring blocks $I_1, I_2, \cdots$ black and white, with lengths increasing fast enough.

Squares and cubes

Corollary

Let $S_k = \{n^k : n \in \mathbb{N}\}$ *be the set of k-powers. Then,* $ord(S_k) \leq 4$.

Problem (1)

What is the order for the set of squares? What is the order for the set of cubes?

For the squares we know that $2 \leq ord(S_2) \leq 3$. For the cubes we know $1 \leq ord(S_3) \leq 3$.

Problem (2)

Is it true that every set $S \subset \mathbb{N}$ *with* $ord(S) = \infty$ *contains an infinite difference set:* $A - A \subset S$ *with* A *an infinite set of integers?*

Bibliography

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