Combinatorial optimization problems

D. Torrão

26th may 2017

This is joint work with: J.C. Rosales and M.B. Branco

Combinatorial optimization problems
• Bracelet Monoids and Numerical Semigroups

> Introduction

> Characterization of the $(n_1, \ldots, n_p)$-bracelets

> The numerical $(n_1, \ldots, n_p)$-bracelets

> The Frobenius variety of the numerical $(n_1, \ldots, n_p)$-bracelet

> Minimal $(n_1, \ldots, n_p)$-system of generators

> Indecomposable $(n_1, \ldots, n_p)$-bracelets

> Sets of positive integers closed under product and the number of decimal digits

Introduction

Introduction

- S - set of segments

Introduction

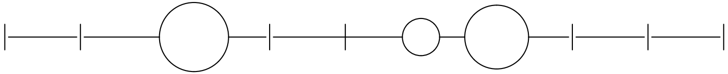
- S - set of segments
- C - set of circles

Introduction

- S - set of segments
- C - set of circles
- (S, C)-bracelet - finite sequence b of the elements in the set $S \cup C$ fulfilling the following conditions:
  1. b does not start by a circle and it does not end by a circle;
  2. in b there are no two consecutive circles.

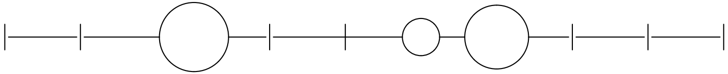
Introduction

- S - set of segments
- C - set of circles
- (S, C)-bracelet - finite sequence b of the elements in the set $S \cup C$ fulfilling the following conditions:
  1. b does not start by a circle and it does not end by a circle;
  2. in b there are no two consecutive circles.



Introduction

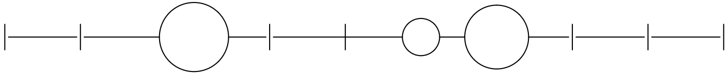
- S - set of segments
- C - set of circles
- (S, C)-bracelet - finite sequence b of the elements in the set $S \cup C$ fulfilling the following conditions:
  1. b does not start by a circle and it does not end by a circle;
  2. in b there are no two consecutive circles.



- $\ell(b)$ - Length of a (S, C)-bracelet b = lengths of all segments + diameters of all circles

Introduction

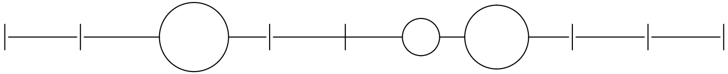
- S - set of segments
- C - set of circles
- (S, C)-bracelet - finite sequence b of the elements in the set $S \cup C$ fulfilling the following conditions:
  1. b does not start by a circle and it does not end by a circle;
  2. in b there are no two consecutive circles.



- $\ell(b)$ - Length of a (S, C)-bracelet b = lengths of all segments + diameters of all circles
- $B(S, C) = \{b \mid b \text{ is a } (S, C) - \text{bracelet}\}$.

Introduction

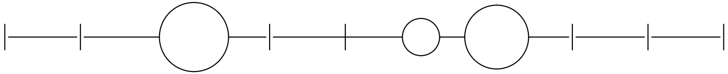
- S - set of segments
- C - set of circles
- (S, C)-bracelet - finite sequence b of the elements in the set $S \cup C$ fulfilling the following conditions:
  1. b does not start by a circle and it does not end by a circle;
  2. in b there are no two consecutive circles.



- $\ell(b)$ - Length of a (S, C)-bracelet b = lengths of all segments + diameters of all circles
- $B(S, C) = \{b \mid b \text{ is a } (S, C) - \text{bracelet}\}$.
- $LB(S, C) = \{\ell(b) \mid b \in B(S, C)\}$

Introduction

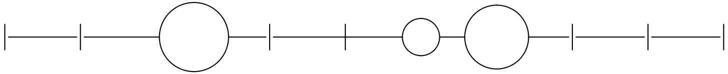
- S - set of segments
- C - set of circles
- (S, C)-bracelet - finite sequence b of the elements in the set $S \cup C$ fulfilling the following conditions:
  1. b does not start by a circle and it does not end by a circle;
  2. in b there are no two consecutive circles.



- $\ell(b)$ - Length of a (S, C)-bracelet b = lengths of all segments + diameters of all circles
- $B(S, C) = \{b \mid b \text{ is a } (S, C) - \text{bracelet}\}$.
- $LB(S, C) = \{\ell(b) \mid b \in B(S, C)\}$
- $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$

Introduction

- S - set of segments
- C - set of circles
- (S, C)-bracelet - finite sequence b of the elements in the set $S \cup C$ fulfilling the following conditions:
  1. b does not start by a circle and it does not end by a circle;
  2. in b there are no two consecutive circles.



- $\ell(b)$ - Length of a (S, C)-bracelet b = lengths of all segments + diameters of all circles
- $B(S, C) = \{b \mid b \text{ is a } (S, C) - \text{bracelet}\}$.
- $LB(S, C) = \{\ell(b) \mid b \in B(S, C)\}$
- $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$
- A submonoid of $(\mathbb{N}, +)$, M, is a $(n_1, \ldots, n_p)$-bracelet if $a + b + \{n_1, \ldots, n_p\} \subseteq M$ for every $a, b \in M \setminus \{0\}$.

Introduction

- A numerical semigroup S is a submonoid of $(\mathbb{N}, +)$ such that $\mathbb{N} \backslash S$ is finite ($gcd(S) = 1$)

Introduction

• A numerical semigroup S is a submonoid of $(\mathbb{N}, +)$ such that $\mathbb{N} \backslash S$ is finite $(gcd(S) = 1)$

• For $A \subset \mathbb{N}$, $< A >= \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N} \backslash \{0\}, x_1, \ldots, x_n \in A$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$

Introduction

- A numerical semigroup S is a submonoid of $(\mathbb{N}, +)$ such that $\mathbb{N}\backslash S$ is finite ($gcd(S) = 1$)
- For $A \subset \mathbb{N}$, $< A >= \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N}\backslash\{0\}, x_1, \ldots, x_n \in A$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$
- The set A is a system os generators of S if $< A >= S$.

Introduction

• A numerical semigroup S is a submonoid of $(\mathbb{N}, +)$ such that $\mathbb{N}\backslash S$ is finite $(gcd(S) = 1)$

• For $A \subset \mathbb{N}$, $< A >= \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N}\backslash\{0\}, x_1, \ldots, x_n \in A$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$

• The set A is a system os generators of S if $< A >= S$.

• A is a minimal system of generators of S if no proper subset of A generates S - $msg(S)$

Introduction

- A numerical semigroup S is a submonoid of $(\mathbb{N}, +)$ such that $\mathbb{N} \backslash S$ is finite $(gcd(S) = 1)$
- For $A \subset \mathbb{N}$, $< A >= \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N} \backslash \{0\}, x_1, \ldots, x_n \in A$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$
- The set A is a system os generators of S if $< A >= S$.
- A is a minimal system of generators of S if no proper subset of A generates S - $msg(S)$
- The largest integer belonging to $\mathbb{Z} \backslash (S)$ is the Frobenius Number of S, denoted by $F(S)$.

Introduction

- A numerical semigroup S is a submonoid of $(\mathbb{N},+)$ such that $\mathbb{N}\backslash S$ is finite $(gcd(S)=1)$
- For $A \subset \mathbb{N}$, $< A >= \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N}\backslash\{0\}, x_1, \ldots, x_n \in A$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$
- The set A is a system os generators of S if $< A >= S$.
- A is a minimal system of generators of S if no proper subset of A generates S - $msg(S)$
- The largest integer belonging to $\mathbb{Z}\backslash(S)$ is the Frobenius Number of S, denoted by $F(S)$.
- PF(S) = $\{x \in \mathbb{Z}\backslash S \mid x + s \in S$ for every $s \in S\backslash\{0\}\} =$
  = set of pseudo-Frobenius numbers of S

Introduction

- A numerical semigroup S is a submonoid of $(\mathbb{N}, +)$ such that $\mathbb{N}\backslash S$ is finite $(gcd(S) = 1)$
- For $A \subset \mathbb{N}$, $< A >= \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N}\backslash\{0\}, x_1, \ldots, x_n \in A$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$
- The set A is a system os generators of S if $< A >= S$.
- A is a minimal system of generators of S if no proper subset of A generates S - $msg(S)$
- The largest integer belonging to $\mathbb{Z}\backslash(S)$ is the Frobenius Number of S, denoted by $F(S)$.
- PF(S) = $\{x \in \mathbb{Z}\backslash S \mid x + s \in S$ for every $s \in S\backslash\{0\}\} =$
  = set of pseudo-Frobenius numbers of S
- SG(S) $= \{x \in$ PF(S) $\mid 2x \in S\}$ = set of special gaps of S

Characterization of the $(n_1, \ldots, n_p)$-bracelets

Proposition

Let $m_1, \ldots, m_q$ and $n_1, \ldots, n_p$ be positive integers and let M be a submonoid of $(\mathbb{N}, +)$ generated by $\{m_1, \ldots, m_q\}$. The following conditions are equivalent.

1. *M is a $(n_1, \ldots, n_p)$-bracelet.*
2. *If $i, j \in \{1, \ldots, q\}$ then $m_i + m_j + \{n_1, \ldots, n_p\} \subseteq M$.*

Characterization of the $(n_1, \ldots, n_p)$-bracelets

Proposition

Let $m_1, \ldots, m_q$ and $n_1, \ldots, n_p$ be positive integers and let M be a submonoid of $(\mathbb{N}, +)$ generated by $\{m_1, \ldots, m_q\}$. The following conditions are equivalent.

1. *M is a $(n_1, \ldots, n_p)$-bracelet.*
2. *If $i, j \in \{1, \ldots, q\}$ then $m_i + m_j + \{n_1, \ldots, n_p\} \subseteq M$.*

Example

Let $M = \langle \{4, 6\} \rangle = \{0, 4, 6, 8, 10, 12, \ldots\}$. We prove that M is a $(2, 4)$-bracelet. As $4 + 4 + \{2, 4\} \subseteq M$, $4 + 6 + \{2, 4\} \subseteq M$ and $6 + 6 + \{2, 4\} \subseteq M$, by applying the previous proposition, we obtain that M is a $(2, 4)$-bracelet.

Given $X \subseteq \mathbb{N}$ we define the $(n_1, \ldots, n_p)$-bracelet generated by X as the intersection of all $(n_1, \ldots, n_p)$-bracelet containing X.

Characterization of the $(n_1, \ldots, n_p)$-bracelets

Proposition

Let $m_1, \ldots, m_q$ and $n_1, \ldots, n_p$ be positive integers and let M be a submonoid of $(\mathbb{N}, +)$ generated by $\{m_1, \ldots, m_q\}$. The following conditions are equivalent.

1. *M is a $(n_1, \ldots, n_p)$-bracelet.*
2. *If $i, j \in \{1, \ldots, q\}$ then $m_i + m_j + \{n_1, \ldots, n_p\} \subseteq M$.*

Example

Let $M = \langle\{4, 6\}\rangle = \{0, 4, 6, 8, 10, 12, \ldots\}$. We prove that M is a $(2, 4)$-bracelet. As $4 + 4 + \{2, 4\} \subseteq M$, $4 + 6 + \{2, 4\} \subseteq M$ and $6 + 6 + \{2, 4\} \subseteq M$, by applying the previous proposition, we obtain that M is a $(2, 4)$-bracelet.

Given $X \subseteq \mathbb{N}$ we define the $(n_1, \ldots, n_p)$-bracelet generated by X as the intersection of all $(n_1, \ldots, n_p)$-bracelet containing X.

- $L_{\{n_1, \ldots, n_p\}}(X)$ is the smallest $(n_1, \ldots, n_p)$-bracelet containing X

Characterization of the $(n_1, \dots, n_p)$-bracelets

Proposition

Let $m_1, \dots, m_q$ and $n_1, \dots, n_p$ be positive integers and let M be a submonoid of $(\mathbb{N}, +)$ generated by $\{m_1, \dots, m_q\}$. The following conditions are equivalent.

1. *M is a $(n_1, \dots, n_p)$-bracelet.*
2. *If $i, j \in \{1, \dots, q\}$ then $m_i + m_j + \{n_1, \dots, n_p\} \subseteq M$.*

Example

Let $M = \langle\{4, 6\}\rangle = \{0, 4, 6, 8, 10, 12, \dots\}$. We prove that M is a $(2, 4)$-bracelet. As $4 + 4 + \{2, 4\} \subseteq M$, $4 + 6 + \{2, 4\} \subseteq M$ and $6 + 6 + \{2, 4\} \subseteq M$, by applying the previous proposition, we obtain that M is a $(2, 4)$-bracelet.

Given $X \subseteq \mathbb{N}$ we define the $(n_1, \dots, n_p)$-bracelet generated by X as the intersection of all $(n_1, \dots, n_p)$-bracelet containing X.

- $L_{\{n_1, \dots, n_p\}}(X)$ is the smallest $(n_1, \dots, n_p)$-bracelet containing X
- If $M = L_{\{n_1, \dots, n_p\}}(X)$ we say that X is a $(n_1, \dots, n_p)$-system of generators of M. Moreover, if no proper subset of X generates M, then we say that X is a minimal $(n_1, \dots, n_p)$-system of generators.

Characterization of the $(n_1, \ldots, n_p)$-bracelets

Theorem

Let $X = \{x_1, \ldots, x_t\} \subseteq \mathbb{N} \setminus \{0\}$ *and let* $\{n_1, \ldots, n_p\} \subseteq \mathbb{N} \setminus \{0\}$*. Then*

$$L_{\{n_1,\ldots,n_p\}}(X) = \{a_1 x_1 + \cdots + a_t x_t + b_1 n_1 + \cdots + b_p n_p \mid a_1, \ldots, a_t, b_1, \ldots, b_p \in \mathbb{N} \text{ and } a_1 + \cdots + a_t > b_1 + \cdots + b_p\} \cup \{0\}.$$

Characterization of the $(n_1, \ldots, n_p)$-bracelets

Theorem

Let $X = \{x_1, \ldots, x_t\} \subseteq \mathbb{N} \setminus \{0\}$ *and let* $\{n_1, \ldots, n_p\} \subseteq \mathbb{N} \setminus \{0\}$. *Then*

$$L_{\{n_1,\ldots,n_p\}}(X) = \{a_1x_1 + \cdots + a_tx_t + b_1n_1 + \cdots + b_pn_p \mid a_1, \ldots, a_t, b_1, \ldots, b_p \in \mathbb{N} \text{ and } a_1 + \cdots + a_t > b_1 + \cdots + b_p\} \cup \{0\}.$$

Example

Let us calculate $L_{\{2,3\}}(\{4\})$. From previous theorem, we have that
$L_{\{2,3\}}(\{4\}) = \{a_1 4 + b_1 2 + b_2 3 \mid a_1, b_1, b_2 \in \mathbb{N} \text{ and } a_1 > b_1 + b_2\} \cup \{0\}$.
Therefore
$L_{\{2,3\}}(\{4\}) = \{0, 4, 8, 10, 11, 12, 14, 15, 16, 17, 18, \rightarrow\} = \langle 4, 10, 11, 17 \rangle$.

The numerical $(n_1, \ldots, n_p)$-bracelets

> Introduction

> Characterization of the $(n_1, \ldots, n_p)$-bracelets

> The numerical $(n_1, \ldots, n_p)$-bracelets

> The Frobenius variety of the numerical $(n_1, \ldots, n_p)$-bracelet

> Minimal $(n_1, \ldots, n_p)$-system of generators

> Indecomposable $(n_1, \ldots, n_p)$-bracelets

> Sets of positive integers closed under product and the number of decimal digits

The numerical $(n_1, \ldots, n_p)$-bracelets

- A numerical $(n_1, \ldots, n_p)$-bracelet is a $(n_1, \ldots, n_p)$-bracelet M such that $\gcd(M) = 1$.

The numerical $(n_1, \ldots, n_p)$-bracelets

- A numerical $(n_1, \ldots, n_p)$-bracelet is a $(n_1, \ldots, n_p)$-bracelet M such that $\gcd(M) = 1$.
- $\mathcal{B}(n_1, \ldots, n_p) = \{M \mid M \text{ is a } (n_1, \ldots, n_p) - \text{bracelet}\}$

The numerical $(n_1, \ldots, n_p)$-bracelets

- A numerical $(n_1, \ldots, n_p)$-bracelet is a $(n_1, \ldots, n_p)$-bracelet M such that $\gcd(M) = 1$.
- $\mathcal{B}(n_1, \ldots, n_p) = \{M \mid M \text{ is a } (n_1, \ldots, n_p) - \text{bracelet}\}$
- $\mathcal{N}(n_1, \ldots, n_p) = \{M \in \mathcal{B}(n_1, \ldots, n_p) \mid M \text{ is a numerical } (n_1, \ldots, n_p) - \text{bracelet}\}$

The numerical $(n_1, \ldots, n_p)$-bracelets

- A numerical $(n_1, \ldots, n_p)$-bracelet is a $(n_1, \ldots, n_p)$-bracelet M such that $\gcd(M) = 1$.
- $\mathcal{B}(n_1, \ldots, n_p) = \{M \mid M \text{ is a } (n_1, \ldots, n_p) - \text{bracelet}\}$
- $\mathcal{N}(n_1, \ldots, n_p) = \{M \in \mathcal{B}(n_1, \ldots, n_p) \mid M \text{ is a numerical } (n_1, \ldots, n_p) - \text{bracelet}\}$

Theorem

Let $n_1, \ldots, n_p$ be positive integers and let D be the set of all positive divisors of $\gcd\{n_1, \ldots, n_p\}$. Then

$$\mathcal{B}(n_1, \ldots, n_p) \setminus \{\{0\}\} = \bigcup_{d \in D} \left\{ dS \mid S \in \mathcal{N}\left(\frac{n_1}{d}, \ldots, \frac{n_p}{d}\right) \right\}.$$

The Frobenius variety of the numerical $(n_1, \ldots, n_p)$-bracelet

The Frobenius variety of the numerical $(n_1, \ldots, n_p)$-bracelet

A Frobenius variety is a nonempty set $\mathcal{V}$ of numerical semigroups fulfilling the following conditions:

1. if S and T are in $\mathcal{V}$, then so is $S \cap T$;
2. if S is in $\mathcal{V}$ and it is not equal to $\mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is in $\mathcal{V}$.

The Frobenius variety of the numerical $(n_1, \ldots, n_p)$-bracelet

A Frobenius variety is a nonempty set $\mathcal{V}$ of numerical semigroups fulfilling the following conditions:

1. if S and T are in $\mathcal{V}$, then so is $S \cap T$;
2. if S is in $\mathcal{V}$ and it is not equal to $\mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is in $\mathcal{V}$.

Proposition

Let $n_1, \ldots, n_p$ be positive integers. Then $\mathcal{N}(n_1, \ldots, n_p)$ is a Frobenius variety.

The Frobenius variety of the numerical $(n_1, \ldots, n_p)$-bracelet

We define the graph $G(\mathcal{N}(n_1, \ldots, n_p))$ as follows:

1. the vertices are the elements of $\mathcal{N}(n_1, \ldots, n_p)$;
2. an element $(S, S') \in \mathcal{N}(n_1, \ldots, n_p) \times \mathcal{N}(n_1, \ldots, n_p)$ is an edge if $S \cup \{\mathrm{F}(S)\} = S'$.

The Frobenius variety of the numerical $(n_1, \ldots, n_p)$-bracelet

We define the graph $G(\mathcal{N}(n_1, \ldots, n_p))$ as follows:

1. the vertices are the elements of $\mathcal{N}(n_1, \ldots, n_p)$;
2. an element $(S, S') \in \mathcal{N}(n_1, \ldots, n_p) \times \mathcal{N}(n_1, \ldots, n_p)$ is an edge if $S \cup \{\mathrm{F}(S)\} = S'$.

Theorem

The graph $G(\mathcal{N}(n_1, \ldots, n_p))$ is a tree rooted in $\mathbb{N}$. Moreover, the descendants of $S \in \mathcal{N}(n_1, \ldots, n_p)$ are the elements of the set $\{S \backslash \{x\} \mid x \in \mathrm{msg}(S),\ x > F(S) \text{ and } S \backslash \{x\} \in \mathcal{N}(n_1, \ldots, n_p)\}$.

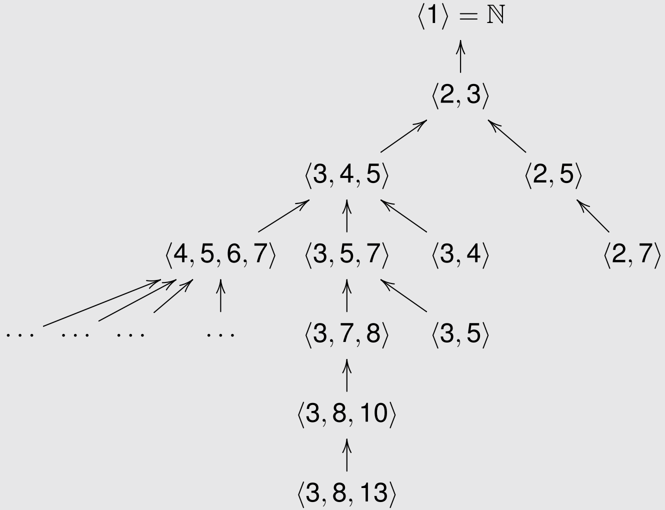
Example

We now draw part of the tree associated to the numerical $(2,3)$-bracelets.

- $\mathbb{N}$ has an only descendant $\mathbb{N}\backslash\{1\} = \langle 2,3\rangle$,
- $\langle 2,3\rangle$ has two descendants $\langle 2,3\rangle\backslash\{2\} = \langle 3,4,5\rangle$ and $\langle 2,3\rangle\backslash\{3\} = \langle 2,5\rangle$,
- $\langle 2,5\rangle$ has an only descendant $\langle 2,5\rangle\backslash\{5\} = \langle 2,7\rangle$,
- $\langle 2,7\rangle$ has no descendants,
- $\langle 3,4,5\rangle$ has three descendants $\langle 3,4,5\rangle\backslash\{3\} = \langle 4,5,6,7\rangle$, $\langle 3,4,5\rangle\backslash\{4\} = \langle 3,5,7\rangle$ and $\langle 3,4,5\rangle\backslash\{5\} = \langle 3,4\rangle$,
- $\langle 3,4\rangle$ has no descendants,
- $\langle 3,5,7\rangle$ has two descendants $\langle 3,5,7\rangle\backslash\{5\} = \langle 3,7,8\rangle$ and $\langle 3,5,7\rangle\backslash\{7\} = \langle 3,5\rangle$,
- $\langle 3,5\rangle$ has no descendants,
- $\langle 3,7,8\rangle$ has an only descendant $\langle 3,7,8\rangle\backslash\{7\} = \langle 3,8,10\rangle$,
- $\langle 3,8,10\rangle$ has an only descendant $\langle 3,8,10\rangle\backslash\{10\} = \langle 3,8,13\rangle$,
- $\langle 3,8,13\rangle$ has no descendants,
- $\langle 4,5,6,7\rangle$ has four descendants

The Frobenius variety of the numerical $(n_1, \ldots, n_p)$-bracelet

Example



Minimal $(n_1, \ldots, n_p)$-system of generators

Proposition

If m is a positive integer, then

$$L_{\{2,3\}}(\{m\}) = \{km + i \mid k \in \mathbb{N}\backslash\{0\},\ i \in \{0, 2, 3, \ldots, 3(k-1)\}\} \cup \{0\}.$$

Minimal $(n_1, \ldots, n_p)$-system of generators

Proposition

If m is a positive integer, then

$$L_{\{2,3\}}(\{m\}) = \{km + i \mid k \in \mathbb{N}\backslash\{0\},\ i \in \{0, 2, 3, \ldots, 3(k-1)\}\} \cup \{0\}.$$

Corollary

If m is a positive integer then $\mathrm{F}\left(L_{\{2,3\}}(\{m\})\right) = \left(\lfloor \frac{m}{3} \rfloor + 2\right) m + 1.$

Minimal $(n_1, \ldots, n_p)$-system of generators

Proposition

If m is a positive integer, then

$$L_{\{2,3\}}(\{m\}) = \{km + i \mid k \in \mathbb{N}\backslash\{0\},\ i \in \{0, 2, 3, \ldots, 3(k-1)\}\} \cup \{0\}.$$

Corollary

If m is a positive integer then $\mathrm{F}\left(L_{\{2,3\}}(\{m\})\right) = \left(\lfloor \frac{m}{3} \rfloor + 2\right) m + 1.$

Example

Let us calculate the set of elements in $L_{\{2,3\}}(\{7\})$. In view of the previous corollary we obtain that $\mathrm{F}\left(L_{\{2,3\}}(\{7\})\right) = 29$. By using the above proposition we have that

$L_{\{2,3\}}(\{7\}) = \{0\} \cup \{7\} \cup (14 + \{0, 2, 3\}) \cup (21 + \{0, 2, 3, 4, 5, 6\}) \cup (28 + \{0, 2, 3, 4, 5, 6, 7, 8, 9\}) \cup \{30, \rightarrow\}$

and thus

$L_{\{2,3\}}(\{7\}) = \{0, 7, 14, 16, 17, 21, 23, 24, 25, 26, 27, 28, 30, \rightarrow\} = \langle 7, 16, 17, 25, 26, 27, 36 \rangle$.

Indecomposable $(n_1, \ldots, n_p)$-bracelets

> Introduction

> Characterization of the $(n_1, \ldots, n_p)$-bracelets

> The numerical $(n_1, \ldots, n_p)$-bracelets

> The Frobenius variety of the numerical $(n_1, \ldots, n_p)$-bracelet

> Minimal $(n_1, \ldots, n_p)$-system of generators

> **Indecomposable $(n_1, \ldots, n_p)$-bracelets**

> Sets of positive integers closed under product and the number of decimal digits

Indecomposable $(n_1, \ldots, n_p)$-bracelets

We say that a $(n_1, \ldots, n_p)$-bracelet is indecomposable if it can not be expressed as an intersection of $(n_1, \ldots, n_p)$-bracelets that contain it properly.

Indecomposable $(n_1, \ldots, n_p)$-bracelets

We say that a $(n_1, \ldots, n_p)$-bracelet is indecomposable if it can not be expressed as an intersection of $(n_1, \ldots, n_p)$-bracelets that contain it properly.

Proposition

Let $m_1, \ldots, m_q$ be positive integers such that $S = \langle m_1, \ldots, m_q \rangle$ is a numerical $(n_1, \ldots, n_p)$-bracelet. Then S is an indecomposable $(n_1, \ldots, n_p)$-bracelet if and only if for every $x \in SG(S) \setminus \{\mathrm{F}(S)\}$ we have that $x + \{x, m_1, \ldots, m_q\} + \{n_1, \ldots, n_p\} \not\subseteq S$.

Indecomposable $(n_1, \ldots, n_p)$-bracelets

We say that a $(n_1, \ldots, n_p)$-bracelet is indecomposable if it can not be expressed as an intersection of $(n_1, \ldots, n_p)$-bracelets that contain it properly.

Proposition

Let $m_1, \ldots, m_q$ be positive integers such that $S = \langle m_1, \ldots, m_q \rangle$ is a numerical $(n_1, \ldots, n_p)$-bracelet. Then S is an indecomposable $(n_1, \ldots, n_p)$-bracelet if and only if for every $x \in SG(S) \setminus \{\mathrm{F}(S)\}$ we have that $x + \{x, m_1, \ldots, m_q\} + \{n_1, \ldots, n_p\} \nsubseteq S$.

Example

• $S = \langle 5, 12, 19, 26, 33 \rangle = \{0, 5, 10, 12, 15, 17, 19, 20, 22, 24, 25, 26, 27, 29, 30, 31, 32, 33, \rightarrow\}$
• $\mathrm{F}(S) = 28$ • $\mathrm{PF}(S) = \{7, 14, 21, 28\}$ • $\mathrm{SG}(S) = \{21, 28\}$ Since $21 + 5 + 2 = 28 \notin S$ We can conclude that the numerical (2)-bracelet $S = \langle 5, 12, 19, 26, 33 \rangle$ is an indecomposable (2)-bracelet

Sets of positive integers closed under product and the number of decimal digits

Sets of positive integers closed under product and the number of decimal digits

- $\ell(n)$ is the number of digits of a positive integer n written in decimal expansion.

Sets of positive integers closed under product and the number of decimal digits

- $\ell(n)$ is the number of digits of a positive integer n written in decimal expansion.

- A digital semigroup D is a subsemigroup of $(\mathbb{N}\backslash\{0\}, \cdot)$ such that if $d \in D$ then $\{x \in \mathbb{N}\backslash\{0\} \mid \ell(x) = \ell(d)\} \subseteq D$.

Sets of positive integers closed under product and the number of decimal digits

- $\ell(n)$ is the number of digits of a positive integer n written in decimal expansion.
- A digital semigroup D is a subsemigroup of $(\mathbb{N}\backslash\{0\}, \cdot)$ such that if $d \in D$ then $\{x \in \mathbb{N}\backslash\{0\} \mid \ell(x) = \ell(d)\} \subseteq D$.
- $L(A) = \{\ell(a) \mid a \in A\}$, for A a subset of $\mathbb{N}\backslash\{0\}$.

Sets of positive integers closed under product and the number of decimal digits

- $\ell(n)$ is the number of digits of a positive integer n written in decimal expansion.
- A digital semigroup D is a subsemigroup of $(\mathbb{N}\backslash\{0\}, \cdot)$ such that if $d \in D$ then $\{x \in \mathbb{N}\backslash\{0\} \mid \ell(x) = \ell(d)\} \subseteq D$.
- $L(A) = \{\ell(a) \mid a \in A\}$, for A a subset of $\mathbb{N}\backslash\{0\}$.

Lemma

If D is a digital semigroup, then $L(D) \cup \{0\}$ is a numerical semigroup.

Sets of positive integers closed under product and the number of decimal digits

- $\ell(n)$ is the number of digits of a positive integer n written in decimal expansion.

- A digital semigroup D is a subsemigroup of $(\mathbb{N}\backslash\{0\}, \cdot)$ such that if $d \in D$ then $\{x \in \mathbb{N}\backslash\{0\} \mid \ell(x) = \ell(d)\} \subseteq D$.

- $L(A) = \{\ell(a) \mid a \in A\}$, for A a subset of $\mathbb{N}\backslash\{0\}$.

Lemma

If D is a digital semigroup, then $L(D) \cup \{0\}$ is a numerical semigroup.

A numerical semigroup S is called LD-semigroup if there exists a digital semigroup D such that $S = L(D) \cup \{0\}$.

Sets of positive integers closed under product and the number of decimal digits

- $\ell(n)$ is the number of digits of a positive integer n written in decimal expansion.
- A digital semigroup D is a subsemigroup of $(\mathbb{N}\backslash\{0\}, \cdot)$ such that if $d \in D$ then $\{x \in \mathbb{N}\backslash\{0\} \mid \ell(x) = \ell(d)\} \subseteq D$.
- $L(A) = \{\ell(a) \mid a \in A\}$, for A a subset of $\mathbb{N}\backslash\{0\}$.

Lemma

If D is a digital semigroup, then $L(D) \cup \{0\}$ is a numerical semigroup.

A numerical semigroup S is called LD-semigroup if there exists a digital semigroup D such that $S = L(D) \cup \{0\}$.

Let S be a LD-semigroup, then $S = L(D) \cup \{0\}$.

Sets of positive integers closed under product and the number of decimal digits

Theorem

Let S be a numerical semigroup. The following conditions are equivalent.

1) *S is a LD-semigroup.*
2) *If $a, b \in S \backslash \{0\}$ then $a + b - 1 \in S$.*

Sets of positive integers closed under product and the number of decimal digits

Theorem

Let S be a numerical semigroup. The following conditions are equivalent.

1) *S is a LD-semigroup.*
2) *If $a, b \in S \setminus \{0\}$ then $a + b - 1 \in S$.*

Let $\mathcal{D} = \{D \mid D \text{ is a digital semigroup}\}$ and let $\mathcal{L} = \{S \mid S \text{ is a LD-semigroup}\}$.

Sets of positive integers closed under product and the number of decimal digits

Theorem

Let S be a numerical semigroup. The following conditions are equivalent.

1) *S is a LD-semigroup.*
2) *If $a, b \in S \setminus \{0\}$ then $a + b - 1 \in S$.*

Let $\mathcal{D} = \{D \mid D \text{ is a digital semigroup}\}$ and let $\mathcal{L} = \{S \mid S \text{ is a LD-semigroup}\}$.

Corollary

The correspondence $\varphi : \mathcal{D} \to \mathcal{L}$, defined by $\varphi(D) = L(D) \cup \{0\}$, is a bijective map. Furthermore its inverse is the map $\theta : \mathcal{L} \to \mathcal{D}$,
$\theta(S) = \{a \in \mathbb{N}\setminus\{0\} \mid \ell(a) \in S\}$.
So,

$$\mathcal{D} = \{\theta(S) \mid S \text{ is a LD-semigroup}\}.$$

Sets of positive integers closed under product and the number of decimal digits

Proposition

Let $\mathcal{L} = \{S \mid S$ is a LD-semigroup$\}$. The set $\mathcal{L}$ is a Frobenius variety.

Proposition

Let $\mathcal{L} = \{S \mid S$ is a LD-semigroup$\}$. The set $\mathcal{L}$ is a Frobenius variety.

Let $\mathcal{L} = \{S \mid S$ is a LD-semigroup$\}$. We define the graph $G(\mathcal{L})$ as the graph whose vertices are the elements of $\mathcal{L}$ and $(S, S') \in \mathcal{L} \times \mathcal{L}$ is an edge if $S' = S \cup \{\mathrm{F}(S)\}$.

Sets of positive integers closed under product and the number of decimal digits

Proposition

Let $\mathcal{L} = \{S \mid S \text{ is a LD-semigroup}\}$. The set $\mathcal{L}$ is a Frobenius variety.

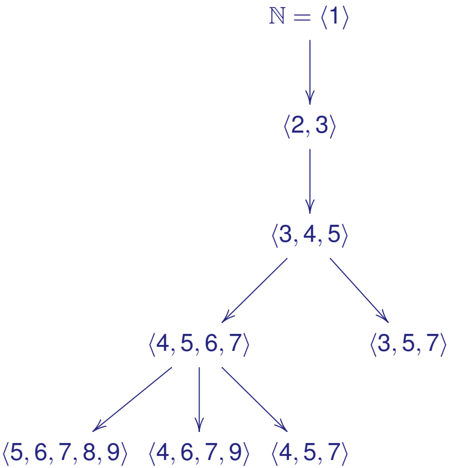
Let $\mathcal{L} = \{S \mid S \text{ is a LD-semigroup}\}$. We define the graph $G(\mathcal{L})$ as the graph whose vertices are the elements of $\mathcal{L}$ and $(S, S') \in \mathcal{L} \times \mathcal{L}$ is an edge if $S' = S \cup \{\mathrm{F}(S)\}$.

Theorem

The graph $G(\mathcal{L})$ is a tree rooted in $\mathbb{N}$. Moreover, the sons of a vertex $S \in \mathcal{L}$ are $S \setminus \{x_1\}, \ldots, S \setminus \{x_l\}$ with
$\{x_1, \ldots, x_l\} = \{x \in \mathrm{msg}(S) \mid x > \mathrm{F}(S) \text{ and } S \setminus \{x\} \in \mathcal{L}\}$

Sets of positive integers closed under product and the number of decimal digits

Figure: The tree of LD-numerical semigroups



Thank you!