Combinatorics of generalized exponents

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I. Generalized exponents in type A

$\mathbb{Z}^n = \bigoplus_{i=1}^{n} \mathbb{Z} e_i$.
Let Q be the sublattice of $\mathbb{Z}^n$ generated by the vectors $e_i - e_{i+1}, 1 \leq i < n$.

For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$, set $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.
The q-Kostant partition function for $\mathfrak{gl}_n(\mathbb{C})$ is defined by

$$\prod_{1 \leq i < j \leq n} \frac{1}{1 - q\frac{x_i}{x_j}} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q(\beta) x^\beta.$$

- $\mathcal{P}_q(\beta) \in \mathbb{Z}_{\geq 0}[q]$

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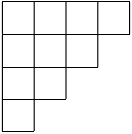
- $\mathcal{P}_q(\beta) \in \mathbb{Z}_{\geq 0}[q]$
- $\mathcal{P}_1(\beta)$ gives the number of nonnegative decompositions of β as a sum of $e_i - e_j, 1 \leq i < j \leq n$.
- $\mathcal{P}_q(\beta) = 0$ when $\beta \notin Q$.

A partition is a sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0) \in \mathbb{Z}^n$.
Each partition is encoded by its Young diagram. For example

$\lambda = (4, 3, 2, 1) \leftrightarrow$  and $|\lambda| = 10$.

Let $\rho = (n, n-1, \ldots, 2, 1)$.

Let $\mathfrak{S}_n$ be the symmetric group of rank n.
The group $\mathfrak{S}_n$ acts on $\mathbb{Z}^n$ by permutation $\sigma \cdot (\beta_1, \ldots, \beta_n) = (\beta_{\sigma(1)}, \ldots, \beta_{\sigma(n)})$.
Consider λ and μ two partitions such that $|\lambda| = |\mu|$.

Definition

The Kostka polynomial $K_{\lambda,\mu}(q)$ is the polynomial of $\mathbb{Z}[q]$ s.t.

$$K_{\lambda,\mu}(q) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \mathcal{P}_q(\sigma(\lambda + \rho) - \rho - \mu)$$

Definition

In the particular case $\mu = \left(\frac{|\lambda|}{n}, \ldots, \frac{|\lambda|}{n}\right)$, the polynomial $K_\lambda(q) = K_{\lambda,\mu}(q)$ is the generalized exponent associated to λ.

A semistandard tableau T of shape λ is a filling of λ by letters in $\{1, \ldots, n\}$ with

- strictly increasing columns from top to bottom
- weakly increasing rows from left to right.

Its weight is $\mathrm{wt}(T) = (\mu_1, \ldots, \mu_n)$ with $\mu_i = \#$ letters i in T

Example

$T =$

1	1	2	4
2	3	5	
3	4		
5			

with weight $\mathrm{wt}(T) = (2, 2, 2, 2, 2)$ and reading

$$\mathrm{w}(T) = 4211532435$$

Theorem

1. $K_{\lambda,\mu}(1)$ *is equal to the number of SST of shape* λ *and weight* μ.
2. $K_{\lambda}(1)$ *is equal to the number of homogeneous SST of shape* λ*, i.e. with weight* $\mu = \left(\frac{|\lambda|}{n}, \dots, \frac{|\lambda|}{n}\right)$.
3. $K_{\lambda}(q) \in \mathbb{Z}_{\geq 0}[q]$.

For Assertion 3 :

- sophisticated geometric or algebraic proofs (Intersection cohomology of nilpotent orbits, affine Kazhdan-Lusztig polynomials, Brylinsky-Kostant filtration).

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- ...which extends in fact to any partition μ (Kostka polynomial).

II. The charge statistics

Examples of row insertions

$$
\begin{array}{|c|c|c|c|}
\hline
1 & 1 & 2 & 4 \\
\hline
2 & 3 & 5 & \multicolumn{1}{c}{} \\
\cline{1-3}
3 & 4 & \multicolumn{2}{c}{} \\
\cline{1-2}
\end{array}
\quad \longleftarrow \quad 5 =
\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 2 & 4 & \mathbf{5} \\
\hline
2 & 3 & 5 & \multicolumn{2}{c}{} \\
\cline{1-3}
3 & 4 & \multicolumn{3}{c}{} \\
\cline{1-2}
\end{array}
$$

$$
\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 2 & 4 & 5 \\
\hline
2 & 5 & 5 & \multicolumn{2}{c}{} \\
\cline{1-3}
3 & 6 & \multicolumn{3}{c}{} \\
\cline{1-2}
\end{array}
\quad \longleftarrow \quad 3 =
\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 2 & \mathbf{3} & 5 \\
\hline
2 & \mathbf{4} & 5 & \multicolumn{2}{c}{} \\
\cline{1-3}
3 & \mathbf{5} & \multicolumn{3}{c}{} \\
\cline{1-2}
\mathbf{6} & \multicolumn{4}{c}{} \\
\cline{1-1}
\end{array}
$$

Cocyclage from T of weight μ:

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 5 \\ \cline{1-3} 3 & 4 \\ \cline{1-2} \mathbf{5} \\ \cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 5 \\ \hline 2 & 3 & 5 \\ \cline{1-3} \mathbf{3} & 4 \\ \cline{1-2}\end{array} \rightarrow \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 \\ \cline{1-3} \mathbf{4} & 5 \\ \cline{1-2}\end{array}$$

$$\rightarrow \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 5 \\ \cline{1-4} 5 \\ \cline{1-1}\end{array} \rightarrow \cdots \rightarrow \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 \\ \hline\end{array}$$

Theorem

Cocyclage operations eventually ends to the unique row R_μ of weight μ

Definition

Set $\mathrm{ch}_n(T) = \|\mu\| - l = \mathrm{ch}_n(R_\mu) - l \geq 0$ where l the number of cocyclage operations needed to get R_μ and

$$\|\mu\| = \sum_{i=1}^{n-1} (n-i)\mu_i.$$

Theorem (LS 1980)

We have

$$K_{\lambda,\mu}(q) = \sum_{T \in SST(\lambda)_\mu} q^{\mathrm{ch}_n(T)}.$$

Example

For $\lambda = (2,1,0)$ and $\mu = (1,1,1)$ we have

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$$

hence

$$\mathrm{ch}_n\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = 1 \text{ and } \mathrm{ch}_n\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = 2$$

thus

$$K_{(2,1,0)}(q) = q + q^2.$$

Let $w = x_1 \cdots x_\ell$ be a word on $\{1 < \cdots < n\}$.
For each $i = 1, \ldots, n-1$, form w_i the subword of w contained only the letters i and $i+1$.
$w_i^{\mathrm{red}} = (i+1)^{\varepsilon_i(w)} i^{\varphi_i(w)}$ is obtained by recursive deletion of factors $i(i+1)$ in w_i.
Example: $w = 2421153243135$ with $n = 5$

1. $w_1 = 21(12)1$ and $w_1^{\mathrm{red}} = 211$. Thus $\varepsilon_1(w) = 1$ and $\varphi_1(w) = 2$
2. $w_2 = 2(23)(23)3$ and $w_2^{\mathrm{red}} = \emptyset$. Thus $\varepsilon_2(w) = \varphi_2(w) = 0$.
3. $w_3 = 4(34)3$ and $w_3^{\mathrm{red}} = 43$. Thus $\varepsilon_3(w) = \varphi_3(w) = 1$.
4. $w_4 = (45)(45)$ and $w_4^{\mathrm{red}} = \emptyset$. Thus $\varepsilon_4(w) = \varphi_4(w) = 0$.

Theorem (LLT 1995)

When T is homogeneous, we have

$$\mathrm{ch}_n(T) = \sum_{i=1}^{n-1} (n-i)\varepsilon_i(\mathrm{w}(T)).$$

Example

For $\lambda = (2,1,0)$ and $\mu = (1,1,1)$ we have

$$\begin{aligned}
\mathrm{ch}_n\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}\right) &= \mathrm{ch}_n(213) = (3-1)\times 1 + (3-2)\times 0 = 2 \\
\mathrm{ch}_n\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}\right) &= \mathrm{ch}_n(312) = (3-1)\times 0 + (3-2)\times 1 = 1
\end{aligned}$$

thus

$$K_{(2,1,0)}(q) = q + q^2.$$

Generalized exponents in type C and beyond

Start from

$$\prod_{1\leq i\leq j\leq n}\frac{1}{1-qx_ix_j}\prod_{1\leq i<j\leq n}\frac{1}{1-q\frac{x_i}{x_j}}=\sum_{\beta\in\mathbb{Z}^n}\mathcal{P}_q^{C_n}(\beta)x^\beta.$$

and replace $\mathfrak{S}_n$ by the group W_n of signed permutations on $\{1,\overline{1},2,\overline{2},\ldots,n,\overline{n}\}$

$$w(x)=y\iff w(\overline{x})=\overline{y}.$$

It acts on $\mathbb{Z}^n$ by

$$w\cdot(\beta_1,\ldots,\beta_n)=(\beta'_1,\ldots,\beta'_n)\text{ with }\beta'_i=\begin{cases}\beta_{w(i)}\text{ if }w(i)>0\\-\beta_{-w(i)},\text{ otherwise}\end{cases}$$

Definition

For any partitions λ and μ

$$K_{\lambda,\mu}^{C_n}(q)=\sum_{w\in W_n}\varepsilon^{C_n}(w)\mathcal{P}_q^{C_n}(w(\lambda+\rho)-\rho-\mu)\text{ and }K_\lambda^{C_n}(q)=K_{\lambda,0}^{C_n}(q)$$

In fact $K_{\lambda}^{C_n}(1)$ gives the dimension of the zero weight space in the irreducible $\mathfrak{sp}_{2n}(\mathbb{C})$-representation indexed by λ. Thus :

- $K_{\lambda}^{C_n}(1) = \#$ King tableaux of shape λ and zero weight,
- $K_{\lambda}^{C_n}(1) = \#$ Kashiwara-Nakashima tableaux of shape λ and zero weight
- $K_{\lambda}^{C_n}(1) = \#$ Littelmann paths of shape λ starting and ending at 0.
- This generalizes to any weight.

Problem

Find a charge for type C_n which proves the positivity of the coefficients.

From Cauchy and Littlewood identities on symmetric functions one gets:

Theorem (C.L., C. Lenart)

1. $$\frac{K_\lambda(q)}{\prod_{i=1}^n(1-q^i)} = \sum_{\gamma \in \mathcal{P}_n} q^{|\gamma|} c^\lambda_{\gamma,\gamma^*}.$$

Here

- $\gamma^* = (0, \gamma_1 - \gamma_2, \ldots, \gamma_1 - \gamma_n)$,
- the $c^\lambda_{\gamma,\gamma^*}$'s are tensor product multiplicities,
- the c^λ_ν's are the branching coefficients appearing in the restrictions $V(\nu) \downarrow^{\mathfrak{gl}_{2n}}_{\mathfrak{sp}_{2n}}$

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2. $$\frac{K_\lambda^{C_n}(q)}{\prod_{i=1}^n (1-q^{2i})} = \sum_{\nu \in 2\mathcal{P}_{2n}} q^{|\nu|/2} c^\lambda_\nu$$

Here

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- the $c^\lambda_{\gamma,\gamma^*}$'s are tensor product multiplicities,
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- The combinatorial description of $K_\lambda(q)$ (for $\mathfrak{gl}_n$) with the ε_i's easily follows from 1.

Example

$$K_{\boxminus}^{C_\infty}(q) = \sum_{k \geq 1} q^{2k} = \frac{q^2}{1-q^2}.$$

- The combinatorial description of $K_\lambda(q)$ (for $\mathfrak{gl}_n$) with the ε_i's easily follows from 1.
- Very more difficult for $\mathfrak{sp}_{2n}$ because c_ν^λ has a simple description only when $\nu_k = 0$ for $k > n$ where

$$c_\nu^\lambda = \sum_{\delta \in \mathcal{P}_n^{\boxminus}} c_{\lambda,\delta}^\nu \, .$$

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- Only gets a description of the formal series $K_\lambda^{C_\infty}(q) = \lim_{n\to+\infty} K_\lambda^{C_n}(q)$.

Example

$$K_{\boxminus}^{C_\infty}(q) = \sum_{k\geq 1} q^{2k} = \frac{q^2}{1-q^2}.$$

A “King tableau” T of shape $\lambda = (\lambda_1, \dots, \lambda_n)$ is a filling of λ by letters of

$$\{1 < 2 < 3 < 4 < \cdots < 2n - 1 < 2n\}$$

s.t.

- T is semistandard,
- The letters in row i are greater to $2i - 1$

Such a tableau is distinguished when we have moreover

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Such a tableau is distinguished when we have moreover

- $\varphi_i(T) = 0$ for any odd i
- $\varepsilon_i(T)$ is even for any odd i.

Theorem (C. L. and C. Lenart)

We have

$$K_\lambda^{C_n}(q) = \sum_{T \text{ distinguished of shape } \lambda} q^{\mathrm{ch}^{C_n}(T)}$$

where

$$\mathrm{ch}^{C_n}(T) = \sum_{i=1}^{2n-1} (2n-i) \left\lceil \frac{\varepsilon_i(T)}{2} \right\rceil .$$

Corollary

1. $K_\lambda^{C_{n+1}}(q) - K_\lambda^{C_n}(q)$ *has nonnegative coefficients.*
2. *Determination of the highest and lowest monomials in* $K_\lambda^{C_n}$.
3. *Simple formulas for particular partitions* λ.

Futures directions

1. Other classical types for finite n.
2. Generalization to any dominant weight $\mu \neq 0$.
3. Connect to a conjectural charge statistics defined on KN tableaux from cyclage operation.
4. Parabolic cases (positivity not even known in general).