On the Dowling and Rhodes matroids

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Joint work with

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Simplicial complexes

(Finite abstract) simplicial complexes (also known as hereditary collections) are structures of the form $\mathcal{H} = (V, H)$, where:

- V is a finite nonempty set;
- $H \subseteq 2^V$ is nonempty and closed under taking subsets.

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They admit a unique (up to homeomorphism) realization as subspaces of an euclidean space, and this provides a topological/geometric viewpoint.

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- $X \subseteq V$ is independent if $X \in H$, otherwise it is dependent
- A minimal dependent subset is a circuit
- Circuits determine the complex: $X \subseteq V$ is independent if and only if it contains no circuit

Matroids

- A simplicial complex $\mathcal{H} = (V, H)$ is a matroid if

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- Simple matroids arise from a geometric lattice (i.e. semimodular and atomistic) L, with atoms of L as points and simplexes determined by the chains in L
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- The set of independent columns of a matrix over a field constitutes a matroid, but not every matroid is field representable

An example: graphic matroids

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- Let $\Gamma = (V, E)$ be a finite undirected graph
- Given $X \subseteq E$, we write $X \in F$ iff X is a forest
- Then $\mathcal{H}(\Gamma) = (E, F)$ is the graphic matroid defined by Γ
- Its circuits are the cycles of Γ

Geometry of finite groups

- For every finite group G and every $n \geq 1$, we can define a matroid $D_n(G)$ (Dowling, early 70s)
- If $m, n \geq 3$, then $D_n(G) \cong D_m(H)$ iff $m = n$ and $G \cong H$

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- Dowling geometries play the role of universal objects in matroid theory (Kahn and Kung 1982)
- They are somewhat analogous to projective geometries, but based on groups instead of fields

Subset, partition, cross-section

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- Let π be a partition of $I \subseteq [n]$
- Given maps $f, h : I \to G$, we write

$$f \sim_\pi h \quad \text{if } f|_{\pi_i} \in G(h|_{\pi_i}) \text{ for each block } \pi_i \text{ of } \pi.$$

- $[f]_\pi$ is the equivalence class of f

The Dowling order

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- The Dowling order is given by refinement and omission of blocks
- That is, given two SPCs $(I, \pi, [f]_\pi)$ and $(J, \tau, [h]_\tau)$, we define $(I, \pi, [f]_\pi) \leq_D (J, \tau, [h]_\tau)$ if:
  1) $J \subseteq I$
  2) every block of τ is a union of blocks of π
  3) if π_i is a block of π contained in J, then $f|_{\pi_i} \in G(h|_{\pi_i})$

The Dowling lattice

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- The Dowling geometries are not in general field representable

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- An $n \times n$ matrix over G is monomial if each row and column contains exactly one element of G and the rest are equal to 0
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- $G \wr S_n$ is the multiplicative group of $n \times n$ monomial matrices over G
- $G \wr PT_n$ is the multiplicative group of $n \times n$ column monomial matrices over G

Connection with semigroups

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Theorem (Margolis, Rhodes and Silva 2017)

The poset of principal left ideals of the monoid $G \wr PT_n$ is a lattice isomorphic to the opposite of the Dowling lattice $Q_n(G)$. Furthermore, the usual action of $G \wr S_n$ on $Q_n(G)$ is equivalent to the action of $G \wr S_n$ considered as the group of units of $G \wr PT_n$ on its lattice of principal left ideals.

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- The Rhodes order is based on containment of sets and partitions:
- That is, $(I, \pi, [f]_\pi) \leq_R (J, \tau, [h]_\tau)$ if
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  1) $I \subseteq J$,
  2) every block of π is contained in a (necessarily unique) block of τ,
  3) $[h|_I]_\pi = [f]_\pi$.
- We denote this poset by $R_n(G)$

The Rhodes lattice

Proposition (MRS 2017)

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(ii) $R_n(G)$ is a lattice if and only if $n = 1$ or G is trivial.

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(ii) $R_n(G)$ is a lattice if and only if $n = 1$ or G is trivial.

- We turn $R_n(G)$ into a lattice $\widehat{R}_n(G)$ by adjoining a top element T
- This lattice is not in general geometric

Also a connection with semigroups

- Let $B_n(G)$ be the Brandt inverse semigroup over $[n]$ with structure group G
- We can describe $B_n(G)$ as the multiplicative semigroup of $n \times n$ matrices over $G \cup \{0\}$ having at most one nonzero entry

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Theorem (MRS 2017)

The lattice of aperiodic inverse subsemigroups of $B_n(G)$ containing 0 is isomorphic to the Rhodes lattice $\widehat{R}_n(G)$

A new concept of independence

- In 2011, Izhakian and Rhodes develop a new concept of independence for boolean matrices
- Independence of columns in $M \in M_{m\times n}(\mathbb{B})$ may be defined using the superboolean semiring $\mathbb{SB} = \{0, 1, 2\}$ but admits an alternative combinatorial description:

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- The column subset C is independent if $M[\_, C]$ contains a square submatrix congruent to some lower unitriangular matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ ? & 1 & 0 & \dots & 0 \\ ? & ? & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & ? & ? & \dots & 1 \end{pmatrix}$$

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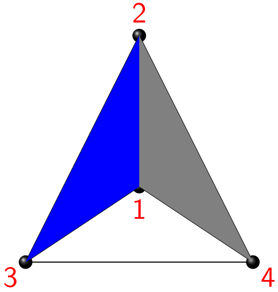
- Every matroid is a BRSC
- The lattice of flats $\mathrm{Fl}(\mathcal{H})$ is atomistic, but needs not be semimodular

Example: T_2

Let $V = 1234$ and $H = P_{\leq 2}(V) \cup \{123, 124\}$. Then

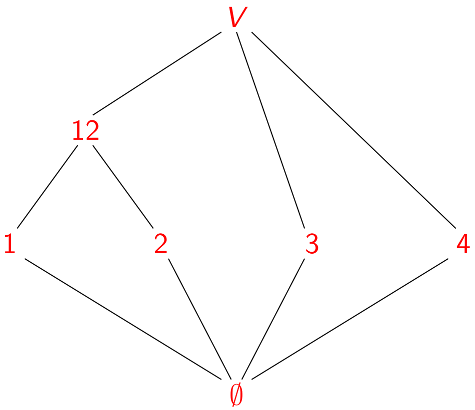
$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

is a boolean matrix representation of $T_2 = (V, H)$:



Example: T_2

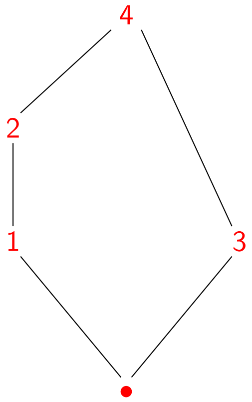
The lattice of flats is



hence not semimodular (T_2 is not a matroid)

Example: T_2

But T_2 is also recognized by the smaller lattice



where the points label a sup-generating set.

A different geometry

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Theorem (MRS 2017)

Let $m, n > 1$ and let G, H be finite nontrivial groups. Then the following conditions are equivalent:

(i) $\widehat{R}_n(G) \cong \widehat{R}_m(H)$;

(ii) $\widehat{\mathcal{H}}_n(G) \cong \widehat{\mathcal{H}}_m(H)$;

(iii) $n = m$ and $G \cong H$.

Gain graphs

- Given a finite graph Γ and a group G, we construct a gain graph by associating elements of G to the edges of Γ with the help of an orientation:
- Given an edge $p \text{ — } q$, we associate a label $g \in G$ to the directed edge $p \longrightarrow q$, and in this case we label the opposite edge $q \longrightarrow p$ by g^{-1}

Balanced cycles

- The label of a (directed) cycle

$$p_1 \xrightarrow{g_1} p_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{m-1}} p_m \xrightarrow{g_m} p_1$$

  is $g_1 \ldots g_m \in G$
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- In particular, the label being 1 does not depend on neither of these factors
- We define as balanced those cycles which have label 1.

The frame matroid

Let Δ be a gain graph. The frame matroid $F(\Delta)$ can be defined by its circuits:

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The lift matroid

The lift matroid $L(\Delta)$ can also be defined by its circuits:

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Defining gain graphs

- $\Delta_n(G)$ is the gain graph obtained from the complete multigraph $|G|K_n$ by attributing all possible labels $g \in G$ to the $|G|$ distinct edges connecting each pair of distinct vertices

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- $\Delta_n(G)$ is the gain graph obtained from the complete multigraph $|G|K_n$ by attributing all possible labels $g \in G$ to the $|G|$ distinct edges connecting each pair of distinct vertices
- $\Delta'_n(G)$ is the gain graph obtained by adjoining to each vertex of $\Delta_n(G)$ a loop labeled by some element $g \in G \setminus \{1\}$

Dowling and frame, Rhodes and lift

Theorem (Zaslavsky 1991)

The Dowling matroid $D_n(G)$ is the frame matroid of the gain graph $\Delta'_n(G)$

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The Dowling matroid $D_n(G)$ is the frame matroid of the gain graph $\Delta'_n(G)$

Theorem (MRS 2017)

The Rhodes matroid $H_n(G)$ is the direct sum of the uniform matroid $U_{n,n}$ with the lift matroid of the gain graph $\Delta_n(G)$

Dowling vs Rhodes

	the Dowling matroid $D_n(1)$	the Rhodes matroid $H_n(1)$
dimension	$n-1$	$2n-2$
size of a minimal lattice representation	2^n	$2^{n-1}+n$
minimum degree of a boolean matrix representation	n	$2n-1$