Expressibility of basic properties of combinatorial polyhedra in FOL and extensions

Jesse Alama

CENTRIA, Universidade Nova de Lisboa, Portugal alama@ftp.unl.pt

Abstract. Consider a first-order relational signature π_3 with a binary relation I and three unary predicate symbols P_0, P_1, and P_2. Intuitively, π_3 is a language for three-dimensional combinatorial polyhedra: P_0, P_1, and P_3 hold, respectively, of the 0-, 1- and 2-polytopes of a polyhedron, and I is an incidence relation for the polytopes. What properties of polyhedra can be expressed in the first-order language π_3? More generally, what properties of n-dimensional combinatorial can be expressed in an appropriate signature π_n (containing n unary predicate symbols P_0, P_1, …, P_{n-1} and a single binary relation I)? These problems, and some natural generalizations, can be solved with basic techniques of finite model theory.

How can we formally express certain properties of combinatorial polyhedra, by which we understand polyhedra considered as incidence structures (as opposed to certain kind of spatial figures or regions)?

Definition 1. *The first-order signature π_3 consists of three unary relation symbols V, E, and F, and one binary relation symbol I.*

What properties of polyhedra can be express using π_3? Can one express, for example, that a finite π_3-structure A satisfies Euler's polyhedron formula, that is, that $|V^A| - |E^A| + |F^A| = 2$? What about the property of being a homology sphere (that is, every cycle is a boundary)? What about the property that $\partial \circ \partial \equiv \emptyset$? And can we express that an π_3-structure comes from a convex three-dimensional polyhedron?

The answer to most of these questions is "no".

Theorem 1. *The properties of (1), being a homology sphere, (2) satisfying Euler's polyhedron formula, (3) satisfying $\partial_k(\partial_{k+1}(c) = \emptyset$ for all $(k+1)$-chains c, (d) being the skeleton of a convex polyhedron are all not expressible by a first-order sentence of the signature π_3.*

Some of the these properties can, however, be expressed with certain extensions of first-order logic, which we shall see.

The aforementioned properties properties are straightforwardly computable: given a finite π_3-structure A, one can obviously compute in a finite amount of time whether A satisfies Euler's polyhedron formula, whether it satisfies the property that $\partial \circ \partial \equiv \emptyset$, and whether it is the skeleton of a convex polyhedron.

(The latter claim is not immediately obvious; one needs to appeal to a basic result known as Steinitz's theorem for that. Steinitz's theorem will be discussed later.) Indeed, it is clear that one can compute most of these properties in time polynomial in the cardinality $|A|$ of the structure A. Fagin's theorem [1] (which says, roughly, that existential second-order logic captures the complexity class NP) then implies that all these properties of finite π_3-structures can be captured in existential second-order logic. This investigation aims to place these properties somewhere between first-order logic and $\exists$-SOL.

(These questions arose from a study of the philosophy of mathematics of Imre Lakatos [2] carried out in the author's dissertation [3].)

References

1. Libkin, L.: Elements of Finite Model Theory. Texts in Theoretical Computer Science. Springer-Verlag (2004)
2. Lakatos, I.: Proofs and Refutations: The Logic of Mathematical Discovery. Cambridge University Press (1976)
3. Alama, J.: Formal Proofs and Refutations. PhD thesis, Stanford University (2009)