Algebraic theory of vector-valued integration

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Abstract. We define a monad $\mathbb{M}$ on the category **BornMeas** of measurable bornological sets, and we show how this monad gives rise to a theory of vector-valued integration that is related to the notion of *Pettis integral*. The monad $\mathbb{M} = (M, e, m)$ associates to each $X \in \mathbf{BornMeas}$ the space MX of all real-valued signed measures of bounded support on X, making $\mathbb{M}$ a variation on the Giry-Lawvere monad of probability measures. Each algebra (X, c) of $\mathbb{M}$ carries the structure of a bornological locally convex topological vector space, and we regard such an algebra as a space on which we can take integrals $\int f d\mu$ of maps $f : T \to X$ with respect to measures $\mu \in MT$. Indeed, we find that the algebraic operations $\Omega^T_\mu : \mathbf{BornMeas}(T, X) \to X$ of arity $T \in \mathbf{BornMeas}$ carried by an algebra (X, c) are naturally construed as operations $f \mapsto \int f d\mu$ of integration with respect to a measure $\mu \in MT$. In fact, we show that a Banach space is an $\mathbb{M}$-algebra as soon as it has a Pettis integral for each incoming bounded weakly-measurable function. It follows that any separable Banach space is an $\mathbb{M}$-algebra, and that any reflexive Banach space, such as any Hilbert space or any $L^p(\mu)$ with $1 < p < \infty$, is an $\mathbb{M}$-algebra.