Structure from sorts, properties, and composition: a minimalist approach to topoalgebraic structure

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Abstract.

We present an alternative approach to the notion of commune introduced at the previous Vancouver CT. We define a ***topoalgebraic category***, or TAC, as a triple (C, S, P) consisting of a locally small category C and sets S and P of objects therein. Objects u of C are interpreted as multisorted algebraic structures consisting of ***elements*** $a : s \to u$ of ***sort*** $s \in S$ and "topologized" with opens or ***states*** $x : u \to p$ for ***property*** $p \in P$. Morphisms $f : u \to v$ of C are interpreted as their left and right actions $(C(S, f), C(f, P))$ on respectively elements of u and states of v. Parallel morphisms with the same actions are called ***equivalent***. A morphism that is both an element and a state is called a ***scalar***, and one that is neither is deemed ***ordinary***.

A TAC is ***didense*** when for any extension by ordinary morphisms alone every new morphism is equivalent to an old one; ***diextensional*** when equivalence is identity; and ***complete*** when for any didense full extension every new object is isomorphic to an old one. A ***community*** is a complete didense diextensional TAC.

Presheaf categories $\widehat{J}$ on a base category J; Chu categories $\mathbf{Chu}(Set, K)$ on a set K; and the Isbell envelope $E(D)$ of a category D, are all instances of communities, defined respectively by $P = \emptyset$ ($S = \mathrm{ob}(J)$); S and P rigid singletons ($K = C(s, p)$); and $S = P$ ($= \mathrm{ob}(D)$). We give examples of useful communities that are none of these, raise the question topos:presheaf-category :: x:community, and report on progress.