# Consequences of Lax Naturality

Art Stone

*Abstract.* Following John Gray [in LNM 391], but using more recent terminology, between strict 2-categories we allow:

- strict 2-functors as 1-cells,
- left-lax natural transformations as 2-cells, and
- modifications as 3-cells.

Even though they do not quite form 3-categories, it turns out that:

0. Left-lax and right-lax natural transformations do not mix well. So, for now, we consider only the left-lax. Gray used right-lax, but the left-lax allow easier examples, *algebraic* examples.
1. The "very horizontal" composite of these 2-cells is a 3-cell (that crosses a certain square of 2-cells.)
2. In a left-lax natural 1-adjunction, the right adjoint turns out to be always *pseudo* natural (that is, natural up to a 3-cell isomorphism.)
3. The "very horizontal" composite of two left lax natural 1-adjunctions produces (what should be called) a *distributive square*. [These were essentially defined by Jon Beck in LNM 80.] And the four possible orientations of the two 1-adjunctions (up-up, down-down, up-down, down-up) produce the four possible types of distributive law, from the resulting distributive squares.

Further, despite not having the full structure of 3-categories, we do get our version of John Gray's *adjunctions between 2-categories* (that is to say, *adjunctions with lax units*), with examples, with their monads and algebras. And, in them, instances of the four types of distributive law.