Von Neumann Categories

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Algebraic Quantum Field Theory (Haag-Kastler)

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- Algebraic quantum field theory (AQFT) is a mathematically rigorous framework for modelling the interaction of quantum mechanics and relativity.
- It is also explicitly category-theoretic; essentially an AQFT is a well-behaved functor.
- We consider Minkowski space as an ordered set with the causal ordering and then take the set of relatively compact opens. These opens form a directed poset under inclusion.

- An AQFT is then a functorial assignment of a C^*-algebra to each interval. So we have a map:

$$U \mapsto \mathcal{A}(U)$$

  The algebras $\mathcal{A}(U)$ are called *local algebras*. They are the algebras of observables local to that region. Then, as in the C^*-algebraic interpretation of QM, a local *state* is then a positive linear functional.

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  The algebras $\mathcal{A}(U)$ are called *local algebras*. They are the algebras of observables local to that region. Then, as in the C^*-algebraic interpretation of QM, a local *state* is then a positive linear functional.

- Since this set of opens in Minkowski space is directed, one can form the directed colimit of the local algebras. It will just be the closure of the union. The result is denoted $\hat{\mathcal{A}}$, and called the *quasilocal algebra*.

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- (Einstein Causality) If U and V are spacelike separated regions, i.e. there can be no causal influence in either direction, then the local algebras $\mathcal{A}(U)$ and $\mathcal{A}(V)$ pairwise commute in the quasilocal algebra.
- One typically adds further conditions, such as invariance with respect to the action of the Poincaré group, but we'll ignore this.

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- The primary examples are Rel, the category of sets and relations, and Hilb_{fd}, the category of finite-dimensional Hilbert spaces.

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- A straightforward modification of the definition of AQFT would be to assign to each interval in spacetime a compact closed dagger category. Much of the above structure is easily lifted to this level.
- The problem is with expressing Einstein Causality.

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- We propose modifying the usual notion of compact closed dagger category by replacing the monoidal structure with premonoidal structure (Power-Robinson).
- The lack of bifunctoriality in the definition of premonoidal category will allow us to express a categorical version of the above causality condition.
- Most C^*-algebras that arise in AQFT are *von Neumann algebras*. The notion of commutant in premonoidal categories allows us to define a categorification of von Neumann algebra.

Premonoidal Categories (Power, Robinson)

Definition

A **binoidal** *category consists of a category* $\mathcal{C}$ *and functors* $H_B : \mathcal{C} \longrightarrow \mathcal{C}$ *and* $K_B : \mathcal{C} \longrightarrow \mathcal{C}$ *for all objects* B *in* $\mathcal{C}$ *and satisfying* $H_B(C) = K_C(B)$ *for all pairs of objects.*

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- In a binoidal category the object $H_B(C) = K_C(B)$ is denoted $B \otimes C$ and for any arrow $f\colon X \longrightarrow Y$ we write $B \otimes f$ for $H_B(f)$ and $f \otimes B$ for $K_B(f)$.

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- Thus in this new notation $H_B = B \otimes -$ and $K_B = - \otimes B$. Notice that $- \otimes -$ is only a functor when one of the arguments is fixed, i.e. it is not a bifunctor.

Definition

A *premonoidal category* consists of a binoidal category $\mathcal{C}$ together with a distinguished object $I \in |\mathcal{C}|$ and natural isomorphisms α, λ and ρ
$\alpha\colon (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$, $\lambda\colon I \otimes A \longrightarrow A$, and $\rho\colon A \otimes I \longrightarrow A$.
These structural isomorphisms must satisfy coherence conditions.

Definition

- If $\mathcal{C}$ is a binoidal category and $f\colon A \longrightarrow C$ is an arrow, then f is *central* if for all arrows $g\colon B \longrightarrow D$,

$$(f \otimes id_B);(id_C \otimes g) = (id_A \otimes g);(f \otimes id_D)$$

  and symmetrically for the two composites $B \otimes A \to D \otimes C$. The results will be denoted $f \otimes g$ and $g \otimes f$.
- More generally, for any f, g as above, we write $f \perp g$ if the above equations hold.
- The *centre* of $\mathcal{C}$ is the category $\mathcal{Z}(\mathcal{C})$ with objects the same as those of $\mathcal{C}$ and its arrows are the central maps in $\mathcal{C}$.

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Proposition

The centre $\mathcal{Z}(\mathcal{C})$ of a premonoidal category $\mathcal{C}$ is a monoidal category.

Example

If M is a monoid, then M is a one object premonoidal category.

Premonoidal Categories IV-Examples

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Example

Let $\mathcal{C}$ be a symmetric monoidal category and let S be a fixed object. Define $\mathcal{C}_S$ as follows: the objects are those of $\mathcal{C}$ and

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There is a canonical functor $I_S \colon \mathcal{C} \to \mathcal{C}_S$, which is the identity on objects and sends f to $f \otimes S$. All maps in the image of this functor are central. Sometimes the converse holds as well.

Theorem

If H is a Hilbert space with dim$\mathrm{H} \geq 1$ then $\mathcal{Z}(\mathbf{Hilb}_{\mathrm{H}}) \simeq \mathbf{Hilb}$.

Premonoidal Quantum Field Theory

Lemma

The directed colimit of a class of symmetric premonoidal dagger categories is a symmetric premonoidal dagger category.

Definition

A *premonoidal quantum field theory* $\mathcal{A}$ is an assignment of a symmetric premonoidal dagger category to each double cone in Minkoswki space. We require the relativistic assumption that if U and V are spacelike separated systems then

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In applications, we will typically assume that the premonoidal categories are all premonoidal dagger closed subcategories of some $\mathbf{Hilb}_{\mathrm{H}}$ especially *von Neumann categories*, to be defined next.

Quote (Rudolf Haag, *Local Quantum Physics*)

From the previous chapters of this book it is evidently not obvious how to achieve a division of the world into parts to which one can assign individuality. . . . Instead we used a division according to regions in space-time. This leads in general to open systems. . .

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In the monoidal approach to modeling quantum processes in spacetime, a process would be modelled by a map $f : A \longrightarrow B$.

An advantage of our premonoidal approach is that we should be able to explicitly model the interaction of the environment on processes. In our setting, a process will be modelled by an arrow $f : A \otimes \mathrm{H} \longrightarrow B \otimes \mathrm{H}$.

The Hilbert space H can be chosen so that it represents states of the entire system, and then the map f will contain information about the interaction of the process with the environment.

Von Neumann Algebras

Definition

Let $\mathcal{B}(\mathrm{H})$ be the C^*-algebra of bounded linear operators on a Hilbert space H. A subset $A \subseteq \mathcal{B}(\mathrm{H})$ closed under the $*$-operation is a *von Neumann algebra* if $A = A''$, where B' is the *commutant* of B, i.e.

$$B' = \{f \in \mathcal{B}(\mathrm{H}) | fb = bf \text{ for all } f \in B\}$$

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Example

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Example

A finite-dimensional $*$*-subalgebra of* $\mathcal{B}(\mathrm{H})$ *closed under 1 is a von Neumann algebra.*

Our intuition is that to categorify the notion of von Neumann algebra, we will replace the C^*-algebra $\mathcal{B}(\mathrm{H})$ with the premonoidal category $\mathbf{Hilb}_{\mathrm{H}}$.

Note that $\mathbf{Hilb}_{\mathrm{H}}$ has a compatible dagger operation, in addition to being premonoidal.

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Definition

Let A be a collection of arrows in the category $\mathbf{Hilb}_{\mathrm{H}}$. Then we define the *commutant* of A, denoted A', to be the collection of all arrows

$$\{g \in \mathrm{Ar}(\mathcal{C}) | f \perp g, \text{ for all } f \in A\}$$

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Lemma

For any dagger-closed class of arrows A, we have A' is a premonoidal dagger subcategory of $\mathbf{Hilb}_{\mathrm{H}}$.

Definition

A symmetric monoidal dagger-subcategory C of $\mathbf{Hilb}_{\mathrm{H}}$ is a *von Neumann category* if $\mathcal{C} = \mathcal{C}''$.

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- The primary example of a von Neumann category we have at the moment is $\mathbf{Hilb_H}$. It remains to find a rich class of examples analogous to the operator-theoretic setting.

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- The primary example of a von Neumann category we have at the moment is $\mathbf{Hilb}_\mathrm{H}$. It remains to find a rich class of examples analogous to the operator-theoretic setting.
- It is ongoing work to examine the equivalent definitions of von Neumann algebra, and determine if they lift to this categorical setting.

Factors

Definition

A *factor* is a von Neumann algebra $\mathcal{A}$ with trivial center, i.e. $\mathcal{Z}(\mathcal{A}) = \{c1_{\mathcal{A}} | c \in \mathbb{C}\}$.

Extending our analogy between von Neumann algebras and von Neumann categories, the analogue of the base field $\mathbb{C}$ is the category **Hilb**.

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Ultimately, we have the goal of extending the notion of *type of a factor* to the categorified setting.

One way to construct von Neumann algebras of various types is the *crossed product*.

Crossed Products

Let $M \subseteq \mathcal{B}(\mathrm{H})$ be a von Neumann algebra and suppose $\alpha\colon G \to \mathrm{Aut}(M)$ be an action of a discrete group G. Define the *crossed product* $G \times_\alpha M$ as follows. First let

$$\tilde{\mathrm{H}} = \{\zeta\colon G \to \mathrm{H} | \ \Sigma_{g \in G} ||\zeta(g)|| < \infty\}$$

Then define maps

$$\pi\colon M \to \mathcal{B}(\tilde{\mathrm{H}}) \qquad \pi(a)(\zeta)(g) = (g^{-1} \cdot a)(\zeta(g))$$

$$\lambda\colon G \to \mathcal{B}(\tilde{\mathrm{H}}) \qquad \lambda(g)(\zeta)(u) = \zeta(g^{-1}u)$$

Then $G \times_\alpha M = [\pi(M) \cup \lambda(G)]'' \subseteq \mathcal{B}(\tilde{\mathrm{H}})$. This satisfies some canonical commutation relations.

Let G be a discrete group, viewed as a premonoidal category. Let $\mathcal{C} \subseteq \mathrm{Hilb_H}$ be a von Neumann category. We define a *G-action on* $\mathcal{C}$ to be a premonoidal functor

$$\alpha \colon G \times \mathcal{C} \to \mathcal{C}$$

So, given an arrow $f\colon K \to K'$ in $\mathcal{C}$ and $g \in G$, we have an action $g \cdot f \colon K \to K'$.
We have a premonoidal functor $\lambda\colon G \to \mathrm{Hilb}_{\tilde{\mathrm{H}}}$ given by the same formula.
(G is a 1-object category, and $Hom_{\mathrm{Hilb}_{\tilde{H}}}(I, I) = \mathcal{B}(\tilde{H})$.)
The functor $\pi\colon \mathcal{C} \to \mathrm{Hilb}_{\tilde{H}}$ should be defined similarly. Then, as above,

$$G \times_\alpha \mathcal{C} = [\pi(\mathcal{C}) \cup \lambda(G)]''$$

Conclusion-Stuff To Do

- Work out details of the cross product construction, and use it to generate new examples.
- Find a premonoidal version of the *Doplicher-Roberts Theorem*:

Theorem

Every compact closed $\mathcal{C}^*$-category (essentially a dagger category with compatible normed structure) is equivalent to the category of finite-dimensional unitary representations of a unique compact group.

- What is the appropriate analogue of compact group in the DR-theorem to obtain premonoidal categories?