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Commutative Algebras in Fibonacci Categories

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Let C be a braided monoidal category with braiding denoted $c_{X,Y} : X \otimes Y \to Y \otimes X$.

If C comes equipped with a natural family of isomorphisms $\theta = \{\theta_X : X \to X | X \in ObC\}$ (satisfying suitable axioms) then C is said to be balanced and the members of the family θ are referred to as ribbon twists.

If every object X of C has a dual object X^* together with unit and counit morphisms then C is said to be rigid (or autonomous).

Recall that a tortile (rigid and balanced) monoidal category is said to be fusion when it is semi-simple k-linear together with a k-linear tensor product, finite dimensional hom-spaces and a finite number of simple objects (up to isomorphism).

A fusion category is called modular when it satisfies a certain non-degeneracy (modularity) condition.

An algebra A in a braided monoidal category C is said to be commutative when $\mu c_{A,A} = \mu$.

A commutative algebra A in a balanced category C is called ribbon if $\theta_A = 1_A$.

A set R is called a fusion rule if its integer span $\mathbb{Z}R$ has the structure of an associative unital ring such that the unit element of $\mathbb{Z}R$ belongs to R and

$$r \cdot s \in \mathbb{Z}_{\geq 0}R$$

for any $r, s \in R$.

We also require R to have a rigidity condition which we formulate as follows:
Equip $\mathbb{Z}R$ with the symmetric bilinear form $(-,-)$ defined by $(r,s) = \delta_{r,s}$ for $r, s \in R$. The condition is then defined to be the existence of an involution $(-)^* : R \to R$ such that

$$(r \cdot s, t) = (s, r^* t) \qquad r, s, t \in R$$

Let C be a semi-simple ridged monoidal category. The set $Irr(C)$ of isomorphism classes of simple objects has the structure of a fusion rule and $\mathbb{Z}Irr(C) = K_0(C)$.

The aim is to describe modular categories with the fusion rule

$$\mathrm{Fib} = \{1, x\} : x^2 = 1 + x$$

We let $\mathcal{F}ib$ be a semi-simple k-linear category with simple objects I and X and tensor product defined as

$$X \otimes X = I \oplus X$$

for which there are two fundamental hom-spaces $\mathcal{F}ib(X^2, X)$ and $\mathcal{F}ib(X^2, I)$. We use a tree / string (of sorts) notation for the two corresponding basis vectors:

We exploit this notational convenience to investigate the categorical properties of $\mathcal{F}ib$.

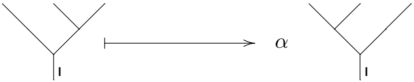
The only non-trivial component of the associativity constraint for $\mathcal{F}ib$ is

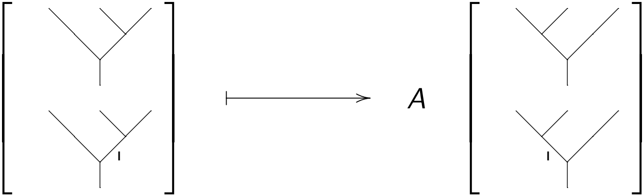
$$\alpha_{X,X,X} : (X \otimes X) \otimes X \to X \otimes (X \otimes X)$$

which on the level of hom-spaces corresponds the two isomorphisms $\mathcal{F}ib(\alpha_{X,X,X}, I)$ and $\mathcal{F}ib(\alpha_{X,X,X}, X)$.

Clearly $dim(\mathcal{F}ib(X^3, I)) = 1$ and $dim(\mathcal{F}ib(X^3, X)) = 2$.
Thus the associativity k-linear transformation for $\mathcal{F}ib(X^3, X)$ is given by $A \in GL_2(k)$ and by $\alpha \in k^*$ for $\mathcal{F}ib(X^3, I)$.

Graphically,



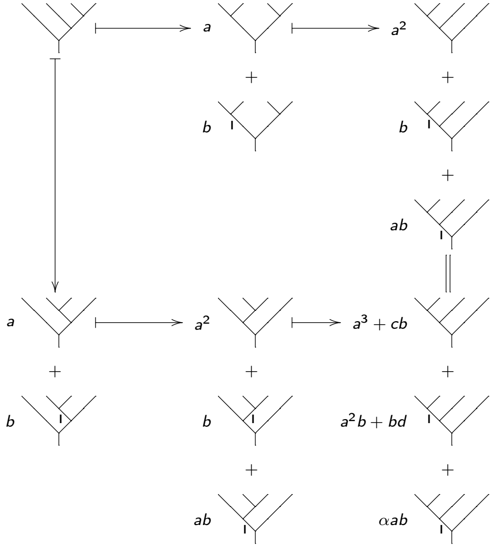


where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The pentagon coherence condition for associativity determines a set of equations relating the entries of the matrix A and α.

Clearly $dim(\mathcal{F}ib(X^4, I)) = 2$ and $dim(\mathcal{F}ib(X^4, X)) = 3$ such that there are five distinct graphical calculations to perform.


a
a^2
+
b
+
ab
a
a^2
$a^3 + cb$
+
b
b
$a^2b + bd$
ab
αab

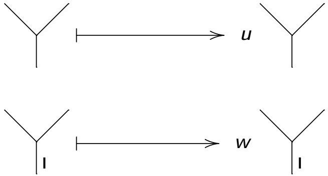
$$\begin{aligned}
\alpha a^2 + bc &= \alpha^2 \\
\alpha ab + bd &= 0 \\
\alpha cb + d^2 &= 1 \\
\alpha ca + dc &= 0 \\
a^3 + bc &= a^2 \\
a^2 b + bd &= b \\
ca^2 + cd &= c \\
abc + d^2 &= 0 \\
\alpha ab &= ab \\
\alpha cb &= d \\
\alpha ca &= ca \\
\alpha^2 d &= cb
\end{aligned}$$

To summarize these calculations, the associativity constraint for $\mathcal{F}ib$ is given by,

$$\alpha = 1, \qquad A = \begin{pmatrix} a & b \\ -ab^{-1} & -a \end{pmatrix}$$

where a is a solution of $a^2 = a + 1$.
We note that $det(A) = -1$ and $A^{-1} = A$.

Similar calculations are performed to obtain a classification of possible braidings and corresponding balanced structures. Braidings come from first considering the form these categorical operations take on the level of fundamental hom-spaces and then running through the axioms.



where $w, u \in k^*$.

It turns out that braidings are completely determined by a number u (where $w = u^2$) satisfying the relation $u^2 = ua - 1$. Together with $a^2 = 1 + a$ these two equations tell us that u is a primitive root of unity of order 10.
Thus the domain of definition for a braided Fibonacci category is the cyclotomic field $\mathbb{Q}(\sqrt[10]{1})$.

Balanced structures require us to consider the endo-hom-spaces of the two simple objects I and X. Since these spaces are one dimensional (basis vectors are the respective identities) the ribbon twists are simply scalar multiples of the basis vectors.

We find that $\theta_1 = id_1$ and $\theta_X = \rho \cdot id_X$ where naturality has forced the scalar multiple corresponding to θ_1 to be unity. The ensuing calculations tells us that balanced structures are completely determined by the braiding as $\rho = u^{-2}$.

By studying possible monoidal equivalences one finds that, up to monoidal equivalence, associativity constraints for Fibonacci categories correspond to solutions of $a^2 = 1 + a$, such that

$$\alpha = 1, \quad A = \begin{pmatrix} a & 1 \\ -a & -a \end{pmatrix}$$

Further more, for each associativity constraint we have that braided balanced structures (up to braided equivalence) correspond to solutions of $u^2 = au - 1$.

The above, taken together with studying rigid structures, yields the following.

Theorem

Every braided balanced structure on a Fibonacci category is modular. Thus there are four non-equivalent Fibonacci modular categories $\mathcal{F}ib_u$, parameterized by primitive roots of unity u of order 10.

A set M is a Non-negative Integer Matrix (NIM) representation of a fusion rule R if $\mathbb{Z}M$ is equipped with the structure of a $\mathbb{Z}R$-module such that

$$r \cdot m \in \mathbb{Z}_{\geq 0}M$$

where $r \in R$ and $m \in M$.
Just as for fusion rules the rigidity condition is imposed on $\mathbb{Z}M$.

Let $\mathcal{M}$ be a semi-simple module category (actegory) over a semi-simple rigid monoidal category C. Then $Irr(\mathcal{M})$ is a NIM-representation of the fusion rule $Irr(C)$.

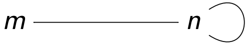
The endgame plan is to study ribbon commutative algebras in $\mathcal{F}ib^{\boxtimes \ell}$ (the tensor powers of $\mathcal{F}ib$). We do this by classifying the NIM-representations of the Fibonacci fusion rule Fib and its tensor powers $\mathrm{Fib}^{\times \ell}$.

It is in this way that we avoid having to directly classify all possible module categories of $\mathcal{F}ib^{\boxtimes \ell}$.

We choose to encode NIM-representations of $\mathrm{Fib}^{\times\ell}$ as a certain type of oriented graph.

Nodes correspond to elements of a NIM-set M. Edges are colored in ℓ colours. Two nodes m and n are the source and the target of an i-th coloured edge respectively iff the multiplicity $(x_i * m, n)$ of n in $x_i * m$ is non-zero. Here $x_i = 1 \otimes \ldots \otimes 1 \otimes X \otimes 1 \otimes \ldots \otimes 1$, where X is in the i-th component.

It turns out that showing there is only one NIM-graph (and so only one irreducible NIM-representation) for Fib is quite straight forward. The graph is

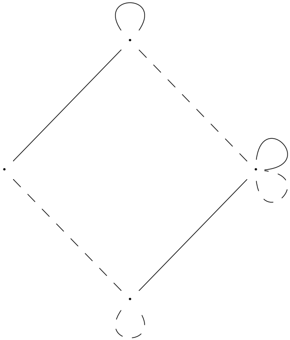


The general case of NIM-representations of $\mathrm{Fib}^{\times\ell}$ is not as easy and requires some fancier footwork.

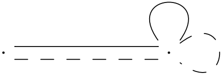
Theorem

Any indecomposable NIM-representation of $Fib^{\times\ell}$ is of the form Fib^{λ} for some set theoretic partition λ of $[\ell] = \{1...\ell\}$.

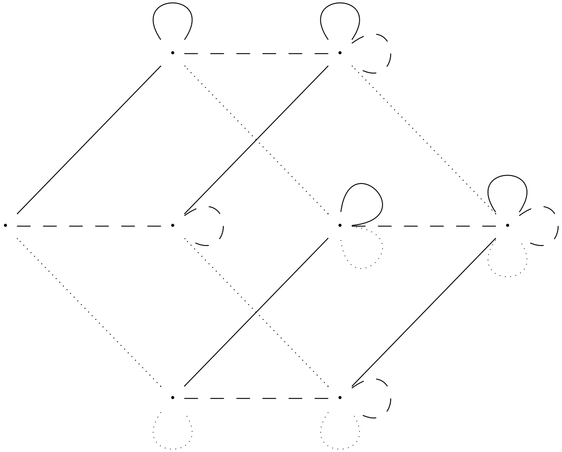
Take for example $\mathrm{Fib}^{\times 2}$. There are two partitions of [2] namely $\{1\} \cup \{2\}$ and $\{1, 2\}$ corresponding to the square



and the double interval



The partition $\{1\} \cup \{2\} \cup \{3\}$ of [3] for $\mathrm{Fib}^{\times 3}$ corresponds to the cube



Let A be an indecomposable algebra in $C = \mathcal{F}ib_u^{\boxtimes \ell}$.
The category C_A of right A modules in C is then an indecomposable (left) C-module category. Recall that the forgetful functor $F : C_A \to C$ (forgetting the module structure) is a morphism of C-module categories and has a left adjoint $G : C \to C_A$ which is also a C-module category morphism. The left adjoint G sends the monoidal unit I to A as module over itself.

It then follows that for the NIM-representation M of C_A we have two maps of NIM-representations (a $\mathbb{Z}R$-module homomorphism): $f : M \to \mathrm{Fib}^{\times \ell}$ and $g : \mathrm{Fib}^{\times \ell} \to M$ which are adjoint in the sense of the rigidity condition

$$(g(y), m)_M = (y, f(m))_{\mathrm{Fib}^{\times \ell}}$$

Since C_A is indecomposable as a C-module category, so is its NIM-representation M.
The afore theorem says we should have $M \simeq \mathrm{Fib}^{\lambda}$ for some set-theoretic partition λ of $[\ell]$.
In the first instance suppose λ has only one part $\lambda = (\ell)$.
In particular $M = \mathrm{Fib}^{(\ell)}$ has just two simple objects: m and n.
Assume that $m = g(1)$.

Since g is a map of NIM-representations we must have $g(x_i) = n$ for all $i = 1, ..., \ell$ such that

$$g(x_i * 1) = x_i g(1) = x_i * m = n$$

Hence for an arbitrary element $x_{i_1} ... x_{i_s}$ of $\mathrm{Fib}^{\times \ell}$

$$g(x_{i_1} ... x_{i_s} * 1) = f_s * n + f_{s-1} * m$$

where f_s is the s-th Fibonacci number.

Thus the adjoint map f has the form

$$f(m) = 1 + \sum_{s=1}^{\ell} f_{s-1} * \sum_{i_1 < \ldots < i_s} x_{i_1} \ldots x_{i_s}$$

$$f(n) = \sum_{s=1}^{\ell} f_s * \sum_{i_1 < \ldots < i_s} x_{i_1} \ldots x_{i_s}$$

We identify $f(m)$ with the class of the algebra A in $K_0(\mathcal{F}ib_u^{\boxtimes\ell}) = \mathbb{Z}[\mathrm{Fib}^{\times\ell}]$. It turns out that since the twist $\theta_{x_{i_1}\dots x_{i_s}} = \theta_{x_{i_1}} \dots \theta_{x_{i_s}}$ of the each of the arbitrary elements depends on s in a non-trivial way, A cannot be ribbon.

Now suppose λ is a set-theoretic partition of $[\ell]$ into ordered parts

$$\lambda = [1...\ell_1][\ell_1...\ell_2]...[\ell_{n-1}...\ell_n]$$

Our theorem says that M, as a NIM-representation of $\mathcal{F}ib_u^{\boxtimes \ell} = \mathcal{F}ib_u^{\boxtimes \ell_1} \boxtimes ... \boxtimes \mathcal{F}ib_u^{\boxtimes \ell_s}$, has the form

$$M = \mathrm{Fib}^{(\ell_1)} \boxtimes ... \boxtimes \mathrm{Fib}^{(\ell_s)}$$

By the baby case above each $\mathbb{Z}[\mathrm{Fib}^{(\ell_i)}] = K_0((\mathcal{F}ib_u^{\boxtimes \ell_i})_{A_i})$ for some non-ribbon algebra $A_i \in \mathcal{F}ib_u^{\boxtimes \ell_i}$.

Then $\mathbb{Z}M = K_0(\boxtimes_{j=1}^{s}(\mathcal{F}ib_u^{\boxtimes \ell_j})_{A_j}) = K_0((\mathcal{F}ib_u^{\boxtimes \ell})_A)$ where $A = \boxtimes_{j=1}^{s} A_i$.

Since A_i are non-ribbon then so is A.

The case for general λ can be reduced to the above by a permutation of $[n]$ which proves the following.

Theorem

There are no (non-trivial) ribbon commutative algebras in $\mathcal{F}ib_u^{\boxtimes \ell}$.

The argument of the proof of the above theorem works well for $\mathcal{F}ib_u^{\boxtimes \ell} \boxtimes \mathcal{F}ib_v^{\boxtimes m}$ as long as $uv \neq 1$.

Thank you!