Differential Categories to Tangential Structure

J.R.B. Cockett

Department of Computer Science
University of Calgary
Alberta, Canada

robin@cpsc.ucalgary.ca

(work with: Robert Seely, Rick Blute,
Goeff Crutwell, and Johnathan Gallagher)

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WHAT IS THIS TALK ABOUT?

Answer: The algebraic/categorical foundations for differential calculus and differential geometry.

$\otimes$-Differential Categories ... (Blute, Cockett, and Seely)

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- $\otimes$-Differential categories = Seely category + differential operator
- Simple abstract framework for differentiation: lots of very sophisticated models
- Inspired by Ehrhard's work: Köthe spaces, finiteness spaces and (with Regnier) on the differential λ-calculus
- Intuition: the "linear algebra" approach to calculus.

$\otimes$-Differential Categories ... (Blute, Cockett, and Seely)

Question: Why are $\otimes$-differential categories not enough?

Answer: Classical differential calculus takes place in a *Cartesian* rather than *linear* (tensor) world .... so (at best) in the coKleisli category of a $\otimes$-differential category ...

Cartesian Differential Categories

Question: what does the coKleisli category of an $\otimes$-differential category look like?

Answer: ... essentially a Cartesian ($\times$-)differential category!

More precisely:
Conjecture: Every $\times$-differential category can be fully embedded into the coKleisli category of a $\otimes$-differential category.

Cartesian Differential Categories

How do we know the axiomatization of Cartesian differential categories is right?

The evidence is mounting!

- CoKleisli categories of (commutative) $\otimes$-differential categories are $\times$-differential categories.
- Sandard examples: smooth maps between $\mathbb{R}^n$, analytic on $\mathbb{C}^n$.
- More later ...

Differential Restriction Categories ... (Cockett, Crutwell, and Gallagher)

Question: Why are $\times$-differential categories still not enough?

Answer: Classical differential calculus considers partial maps and uses topological notions ....

WHAT DO PARTIAL MAP CATEGORIES WITH DIFFERENTIATION LOOK LIKE?

Answer: differential restriction categories!

Tangential Structure ...
(.. mostly Geoff Cruttwell)

Why are differential restriction categories *still* not enough?

Answer: Differential goemetry considers partial maps between differential manifolds....

WHAT DO MANIFOLD CATEGORIES OF DIFFERENTIAL CATEGORIES LOOK LIKE?

Answer: restriction categories with tangential structure!

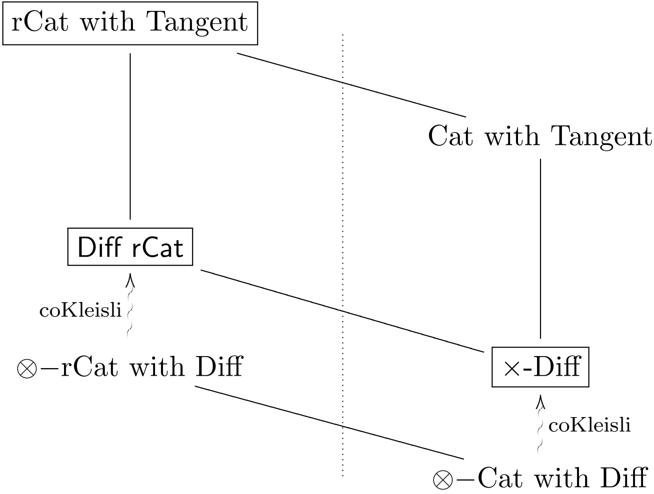
Tangential Structure ...

How do you know you have got this axiomatization right?

The evidence is mounting!!!

But verdict still out ....

The story so far ...



... everything now depends heavily on the notion of $\times$-differential category!

... just write down the equations governing the coKleisli category of a $\otimes$-differential category...

HOW HARD CAN THAT BE?

Well after three years we still had not got it right!

WHY?

A. We were idiots?
B. Academic baggage ...
C. Calculus for the masses ...
D. The structure of the area has been trampled on with:
    - Preconceptions: infinitesimals and dx ....
    - Manipulations without algebraic basis ...
    - Notational short-cuts masks structure ...
E. The axioms are actually quite tricky!

The good news:

We have the *basic* axiomatization right!

FINALLY!

... and people are beginning to use it!

How do we know?

One answer: many people have now reached the same spot wearing different shoes ...

Another answer: sits at the cross roads for many theoretical threads ...

- Faa Di Bruno (cartesian differential categories are coalgebras for a comonad).
- Semantics for the *differential* λ*-calculus*.
- Semantics of the *resource calculus*.

The bad news:

... JUST THE BEGINNING OF THE STORY!!!

Defining Cartesian Differential Categories

Formulation of cartesian differential categories needs:

(a) Left additive categories

(b) Cartesian structure in the presence of left additive structure

(c) Differential structure

Left additive ...

A category $\mathbb{X}$ is a **left-additive category** in case:

- Each hom-set is a commutative monoid $(0,+)$
- $f(g+h) = (fg) + (fh)$ and $f0 = 0$.

$$A \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C$$

A map h is said to be **additive** if it also preserves the additive structure on the right $(f+g)h = (fh) + (gh)$ and $0h = 0$.

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{h} C$$

NOTE: (a) *No* negatives (b) additive maps are the exception but form a subcategory $\mathbb{X}^+$...

Example

(i) The category whose objects are commutative monoids CMon but whose maps need not preserve the additive structure.

(ii) Real vector spaces with smooth maps.

(iii) The coKleisli category for a comonad on an additive category where the comonad is *not* enriched.

Products in left additive categories

A **Cartesian left-additive category** is a left-additive category with products such that:

- the maps π_0, π_1, and Δ are additive;
- pairing preserves additivity.

Lemma

The following are equivalent:

(i) A Cartesian left-additive category;

(ii) A left-additive category for which $\mathbb{X}_+$ has biproducts and the the inclusion $\mathcal{I} : \mathbb{X}_+ \longrightarrow \mathbb{X}$ creates products;

(iii) A Cartesian category $\mathbb{X}$ in which each object is a commutative monoid $(+_A : A \times A \longrightarrow A, 0_A : 1 \longrightarrow A)$ such that $+_{A\times B} = \langle(\pi_0 \times \pi_0)+_A, (\pi_1 \times \pi_1)+_B\rangle$ and $0_{A\times B} = \langle 0_A, 0_B\rangle$.

Lemma

In a Cartesian left-additive category:

(i) *f is additive if and only if*

$$(\pi_0 + \pi_1)f = \pi_0 f + \pi_1 f : A \times A \longrightarrow B \quad \textit{and} \quad 0f = 0 : 1 \longrightarrow B;$$

(ii) *$g : A \times X \longrightarrow B$ is additive in its second argument iff*

$$1 \times (\pi_0 + \pi_1)g = (1 \times \pi_0)g + (1 \times \pi_1)g : A \times X \times X \longrightarrow B$$

$$\textit{and} \quad (1 \times 0)g = 0 : A \times 1 \longrightarrow B.$$

"Multi-additive maps" are maps additive in each argument.

All our earlier example are Cartesian left-additive categories!

An operator $D_\times$ on the maps of a Cartesian left-additive category

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow[D_\times[f]]{} Y}$$

is a **Cartesian differential operator** in case it satisfies:

[CD.1] $D_\times[f+g] = D_\times[f] + D_\times[g]$ and $D_\times[0] = 0$;

[CD.2] $\langle (h+k), v \rangle D_\times[f] = \langle h, v \rangle D_\times[f] + \langle k, v \rangle D_\times[f]$;

[CD.3] $D_\times[1] = \pi_0$, $D_\times[\pi_0] = \pi_0\pi_0$, and $D_\times[\pi_1] = \pi_0\pi_1$;

[CD.4] $D_\times[\langle f, g \rangle] = \langle D_\times[f], D_\times[g] \rangle$ (and $D_\times[\langle\rangle] = \langle\rangle$);

[CD.5] $D_\times[fg] = \langle D_\times[f], \pi_1 f \rangle D_\times[g]$.

[CD.6] $\langle \langle f, 0 \rangle, \langle h, g \rangle \rangle D_\times[D_\times[f]] = \langle f, h \rangle D_\times[f]$;

[CD.7] $\langle \langle 0, f \rangle, \langle g, h \rangle \rangle D_\times[D_\times[f]] = \langle \langle 0, g \rangle, \langle f, h \rangle \rangle D_\times[D_\times[f]]$

A Cartesian left-additive category with such a differential operator is a **Cartesian differential category**.

What was so hard about that?

ANSWER: the last two rules!!

- They are independent ...
- They involve higher differentials ...
- Not so obvious where they come from or why they are needed ...

[CD.1] $D_\times[f+g] = D_\times[f] + D_\times[g]$ and $D_\times[0] = 0$;
(operator preserves additive structure)

[CD.2] $\langle (h+k), v\rangle D_\times[f] = \langle h, v\rangle D_\times[f] + \langle k, v\rangle D_\times[f]$
(always additive in first argument);

[CD.3] $D_\times[1] = \pi_0$, $D_\times[\pi_0] = \pi_0\pi_0$, and $D_\times[\pi_1] = \pi_0\pi_1$
(coherence maps are linear -differential constant);

[CD.4] $D_\times[\langle f, g\rangle] = \langle D_\times[f], D_\times[g]\rangle$ (and $D_\times[\langle\rangle] = \langle\rangle$)
(operator preserves pairing);

[CD.5] $D_\times[fg] = \langle D_\times[f], \pi_1 f\rangle D_\times[g]$ (chain rule);

[CD.6] $\langle\langle f, 0\rangle, \langle h, g\rangle\rangle D_\times[D_\times[f]] = \langle f, h\rangle D_\times[f]$
(differentials are linear[1] in first argument);

[CD.7] $\langle\langle 0, f\rangle, \langle g, h\rangle\rangle D_\times[D_\times[f]] = \langle\langle 0, g\rangle, \langle f, h\rangle\rangle D_\times[D_\times[f]]$
(partial differentials commute);

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[1] In the sense of the differential being constant.

Real vector spaces with smooth maps are the "standard" example of a Cartesian differential category.

$$\frac{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} f_1(x_1, .., x_n) \\ \vdots \\ f_m(x_1, .., x_n) \end{pmatrix}}{\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}\right) \mapsto \begin{pmatrix} \frac{\mathrm{d}f_1(\tilde{x})}{\mathrm{d}x_1}(x_1) \cdot u_1 + \ldots + \frac{\mathrm{d}f_1(\tilde{x})}{\mathrm{d}x_n}(x_n) \cdot u_n \\ \vdots \\ \frac{\mathrm{d}f_m(\tilde{x})}{\mathrm{d}x_1}(x_1) \cdot u_1 + \ldots + \frac{\mathrm{d}f_m(\tilde{x})}{\mathrm{d}x_n}(x_n) \cdot u_n \end{pmatrix}} \; D$$

RECALL: Cartesian differential categories have no partial maps ...

Introductory analysis uses partial maps and topological notions
.... functions which are "locally" differentiable

NEED TO ADD PARTIALITY

Restriction categories

A convenient algebraic formulation for partiality ....

A **restriction category** is a category $\mathbb{X}$ with a combinator, $\overline{(\ )} : \mathbb{X}(A,B) \longrightarrow \mathbb{X}(A,A)$, satisfying

[R.1] $\overline{f} f = f$;

[R.2] $\overline{f}\ \overline{g} = \overline{g}\ \overline{f}$;

[R.3] $\overline{f}\ \overline{g} = \overline{\overline{f} g}$;

[R.4] $f \overline{h} = \overline{fh} f$.

A map is **total** when $\overline{f} = 1$.

Restriction category are *precisely* full subcategory of partial map categories ...

Examples of restriction categories

Example

(i) Par the category of sets and partial functions: $\overline{f}$ gives the domain of definition of f.

$$\overline{f}(x) = \begin{cases} x & f(x)\downarrow \\ \uparrow & \text{else} \end{cases}$$

(ii) $\text{Top}_{\mathcal{O}}$ the category of topological spaces and continuous maps defined on open sets: $\overline{f}$ is the open set on which the function is defined.

(iii) $\text{Par}(\text{CRing}^{\text{op}}, \text{Loc})$ the category of commutative rings opposite with "rational" maps.

QUESTION: Does this last have differential structure?

Defining Differential Restriction Categories

The steps ...

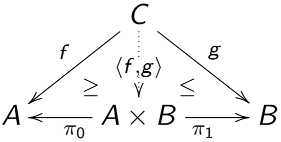
(a) *Cartesian* restriction categories,

(b) Cartesian *left additive* restriction categories,

(c) *Differential structure*.

Cartesian Restriction Categories

Restriction category are partial order enriched by $f \leq g \Leftrightarrow \overline{f}g = f$.
A **restriction product** of A, B is an object $A \times B$ such that for any $f : C \longrightarrow A$ and $g : C \longrightarrow B$ there is a unique pairing map $\langle f, g \rangle : C \longrightarrow A \times B$ such that



where π_0, π_1 are total and $\overline{\langle f, g \rangle} = \overline{f}\ \overline{g}$.
A **restriction terminal object** is $\mathbf{1}$ such that for any object A, there is a unique total map $!_A : A \longrightarrow \mathbf{1}$ which satisfies $!_{\mathbf{1}} = 1_{\mathbf{1}}$.
Further, for any map $f : A \longrightarrow B$, $f!_B \leq !_A$.
A restriction category with all restriction products is **Cartesian**.

A **left additive restriction category** has each $\mathbb{X}(A, B)$ a commutative monoid with $\overline{f + g} = \overline{f}\,\overline{g}$ and 0 being total. Furthermore, $h(f + g) = hf + hg$ and $s0 = \overline{s}0$
A map, h, in a left additive restriction category is a **total additive** if h is total, and $(f + g)h = fh + gh$.
A **cartesian left additive restriction category** is both a left additive restriction category and a cartesian restriction category where $\pi_0, \pi_1,$ and Δ are total additives, and $(f + h) \times (g + k) = (f \times g) + (h \times k)$.

A **differential restriction category** is a cartesian left additive restriction category with a differential combinator

$$\frac{f : X \longrightarrow Y}{D[f] : X \times X \longrightarrow Y}$$

such that

[DR.1] $D[f+g] = D[f] + D[g]$ and $D[0] = 0$ **(additivity)**;

[DR.2] $\langle g+h, k\rangle D[f] = \langle g, k\rangle D[f] + \langle h, k\rangle D[f]$ and $\langle 0, g\rangle D[f] = \overline{gf}0$;

[DR.3] $D[1] = \pi_0$, $D[\pi_0] = \pi_0\pi_0$, and $D[\pi_1] = \pi_0\pi_1$;

[DR.4]] $D[\langle f, g\rangle] = \langle D[f], D[g]\rangle$;

[DR.5] $D[fg] = \langle D[f], \pi_1 f\rangle D[g]$ **(chain rule)**;

[DR.6] $\langle\langle g, 0\rangle, \langle h, k\rangle\rangle D[D[f]] = \overline{h}\langle g, k\rangle D[f]$ **(linearity)**

[DR.7] $\langle\langle 0, h\rangle, \langle g, k\rangle\rangle D[D[f]] = \langle\langle 0, g\rangle, \langle h, k\rangle\rangle D[D[f]]$;

[DR.8] $D[\overline{f}] = (1 \times \overline{f})\pi_0$; **(differential of restriction)**

[DR.9] $\overline{D[f]} = 1 \times \overline{f}$ **(restriction of differential)**.

Example

(i) Smooth Maps on open subsets of $\mathbb{R}^n$

(ii) Rational Functions over any ring ....

$$\mathsf{Rat}(R) \subseteq \mathsf{Par}(\mathsf{CRing}_R^{\mathrm{op}}, \mathsf{Loc})$$

(full subcategory of polynomial rings $R[x_1, ..., x_n]$ over R)

Jacobian matrix provides the differential structure in both cases ...
The second example does *not* have the differential given by limits in the topological as it relies on the formal derivative ....

Manifold completion to tangent structure

(a) From any differential restriction $\mathbb{X}$ category we can *join complete* to obtain a differential join restriction category $\text{join}(\mathbb{X})$.
Join structure always works well with differentail structure.

(b) Given a differential join restriction category $\text{join}(\mathbb{X})$ we can form the *manifold completion* $\text{Man}(\text{join}(\mathbb{X}))$.

(c) $\text{Man}(\text{join}(\mathbb{X}))$ is *not* a differential restriction category but it has *tangent structure* .... which axiomatizes the categories arising in differential geometry.

Joins ..

In a restriction category, parallel maps f and g are **compatible** if $\overline{f}g = \overline{g}f$.

A restriction category, $\mathbb{X}$, is a **join restriction category** if every set of compatible maps, $C \subseteq \mathbb{X}(A, B)$, has a join (sup) that is stable; i.e.,

$$f\left(\bigvee_{g \in C} g\right) = \bigvee_{g \in C} fg.$$

Theorem

Join and differential restriction structure are compatible; i.e.

$$D\left[\bigvee_i f_i\right] = \bigvee D[f_i].$$

Smooth functions are a differential join restriction category, but rational functions are not.
Rational functions often has a sup for a set of compatible maps but stability fails:

$$(1, \langle x-1 \rangle) \smile (1, \langle y-1 \rangle),$$

so the join must be

$$(1, \langle 1 \rangle).$$

As a counterexample, consider the substitution $\left[x^2/x, x^2/y\right]$. $\langle x-1 \rangle \cap \langle y-1 \rangle$ does not contain x or y; thus, the substitution does not contain $x-1$. However,

$$x - 1 \in \left(\left[x^2/x, x^2/y\right] \langle x-1 \rangle \cap \left[x^2/x, x^2/y\right] \langle y-1 \rangle\right).$$

Join Completion

Let $\mathbb{X}$ be any restriction category. We can obtain a join restriction category from $\mathbb{X}$ by a universal construction join($\mathbb{X}$):

Obj: Those of $\mathbb{X}$.

Arr: $A \xrightarrow{\mathcal{F}} B$ is a subset $\mathcal{F} \subseteq \mathbb{X}(A, B)$ that is pairwise compatible and has the property that if $f \in \mathcal{F}$ and $h \leq f$ (i.e. $\overline{h}f = h$) then $h \in \mathcal{F}$.

Id: $\downarrow 1_A = \{d : A \longrightarrow A \mid d \leq 1_A\}$

Comp: $\mathcal{F}\mathcal{G} = \{fg \mid f \in \mathcal{F}, g \in \mathcal{G}\}$

Rest: $\overline{\mathcal{F}} = \{\overline{f} \mid f \in \mathcal{F}\}$

Join: $\bigvee_i \mathcal{F}_i = \bigcup_i \mathcal{F}_i$

Theorem

If $\mathbb{X}$ is a differential restriction category, then join$(\mathbb{X})$ *is a differential join restriction category.*

The differential structure on join$(\mathbb{X})$ is

$$\begin{aligned} D[\mathcal{F}] &= \downarrow \{D[f] \mid f \in \mathcal{F}\} \\ &= \{e \mid e \leq D[f] \text{ for some } f \in \mathcal{F}\} \end{aligned}$$

- This differential restriction structure is not given by a Jacobian.
- We can obtain a differential join restriction category from Rat(R).

Manifolds and atlases ...

Definition

(Grandis) If $\mathbb{X}$ is a join restriction category, an **atlas of objects from** $\mathbb{X}$ consists of a set of objects $X_i \in \mathbb{X}$, together with a series of maps $X_i \xrightarrow{\phi_{ij}} X_j$ such that:

1. $\phi_{ii}\phi_{ij} = \phi_{ij}$;
2. $\phi_{ij}\phi_{jk} \leq \phi_{ik}$;
3. ϕ_{ij} has partial inverse ϕ_{ji}.

Each map $\phi_{ii} : X_i \longrightarrow X_i$ is a restriction idempotent, and represents the "open subset" of X_i that the chart is using. The maps ϕ_{ij} define how these charts overlap.

There is a natural notion of morphism of these:

Definition

If (U_i, ϕ_{ij}) and (V_k, ψ_{kh}) are atlases of $\mathbb{X}$, then an atlas morphism A consists of a family of maps $U_i \xrightarrow{A_{ik}} V_k$ such that:

1. $\phi_{ii} A_{ik} = A_{ik}$;
2. $\phi_{ij} A_{jk} \leq A_{ik}$;
3. $A_{ik} \psi_{kh} = \overline{A_{ik}} A_{ih}$

Composition of atlas morphisms is given by:

$$(AB)_{im} := \bigvee_k A_{ik} B_{km}.$$

For any join restriction category $\mathbb{X}$, we then have a join restriction category $\text{Man}(\mathbb{X})$, with objects atlases, and morphisms atlas morphisms.

Example

(i) Atl(**fdCts**) “is” real topological manifolds;

(ii) Atl(**fdSmooth**) “is” smooth real manifolds;

(iii) Atl($\text{join}_\emptyset(\mathbf{cRing}^{\text{op}}, \text{loc})$) “is” schemes.

Problem: The manifold completion of a differential join restriction category is *NOT* a differential category ...

Tangent structure

Definition

Suppose that $\mathbb{X}$ is a cartesian restriction category. A **tangent structure** for $\mathbb{X}$ consists of:

- (**tangent bundles**) a cartesian restriction functor $\mathbb{X} \xrightarrow{T} \mathbb{X}$ with a total natural transformation $T \xrightarrow{p} I$;
- (**tangent bundle products**) for each $M \in \mathbb{X}$ and natural number $n \geq 2$, the restriction pullback of n copies of $TM \xrightarrow{p} M$ exists and is denoted by $(T_n(M), p_1 \ldots p_n)$, with each p_i total: this implies each T_n is a join restriction functor when there are joins, and the arrows $T_n \xrightarrow{p_i} T$ are natural;

- (**addition of tangent vectors**) there are total natural transformations $T_2 \xrightarrow{+} T$ and $I \xrightarrow{0} T$ such that $+p = p_1 p = p_2 p$ and $0p = 1$, and
  - addition is associative; that is,

$$\begin{array}{ccc} T_3 & \xrightarrow{+_l} & T_2 \\ {\scriptstyle +_r}\downarrow & & \downarrow{\scriptstyle +} \\ T_2 & \xrightarrow[+]{} & T \end{array}$$

    commutes;
  - addition is commutative; that is, $s+ = +$, where s is the map $T_2 \longrightarrow T_2$ induced by reversing p_1 and p_2;
  - addition is unital, so that $T \xrightarrow{z} T_2 \xrightarrow{+} T = 1_T$ (where z is induced by the maps $TM \xrightarrow{0p,1} TM$;

- (**vertical lift**) there is a total natural transformation $T_2 \xrightarrow{l} T^2$ such that $lp_T = p_2$ and $lT(p) = p_2p0$;
- (**canonical flip**) there is a total natural transformation $T^2 \xrightarrow{c} T^2$ such that $cp_T = T(p)$ and $c^2 = 1$ (implies $cT(p) = p_T$);
- (**preservation of addition and zero**) the following diagram commutes:

$$\begin{array}{ccc} TT_2 & \xrightarrow{w} & T_2T \\ {\scriptstyle T(+)}\downarrow & & \downarrow{\scriptstyle +_T} \\ T^2 & \xrightarrow[c]{} & T^2 \end{array}$$

(better as $T(+) = (w)(+_T)(c)$?)where w is the map induced by the universal property of T_2T applied to $T(p_2)c$ and $T(p_1)c$, and $T(0) = 0c$.

Theorem

Any differential restriction category has tangent structure, with $TM = M \times M$, $Tf = \langle Df, \pi_1 f \rangle$, and $T_n, +, 0, c, l$ the obvious maps. Conversely, if a left additive cartesian restriction category has tangent structure such that

- $TM = M \times M$
- T_n *is the canonical pullback;*
- $p, +, 0, c,$ *and* l *are the obvious maps.*

then its tangent structure is equivalent to a differential restriction structure, with $Df := Tf\pi_0$.

Theorem

If $\mathbb{X}$ has differential restriction structure, then $\mathsf{Mf}(\mathbb{X})$ *has tangent structure.*

But there is another important construction .....

Theorem

The (restriction) equalizer completion $\mathsf{Eq}(\mathbb{X})$ *of a differential restriction category $\mathbb{X}$ has tangential structure.*

We believe (strongly)

$$\mathsf{Eq}(\mathsf{Rat}(R)) \equiv \mathsf{Par}(\mathsf{CRing}^{\mathrm{op}}, \mathsf{Loc})$$

so this would mean $\mathsf{Par}(\mathsf{CRing}^{\mathrm{op}}, \mathsf{Loc})$ has tangent structure ....

Question: is this well known?

END