Towards Noncommutative Gel'fand Duality

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Andreas Döring

Department of Computer Science
University of Oxford

andreas.doering@cs.ox.ac.uk

"Those who do not understand the nature of sin and virtue are attached to duality; they wander around deluded."

Sri Guru Granth Sahib

Introduction

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- To each commutative algebra $\mathcal{A}$, the set of algebra homomorphisms $\lambda : \mathcal{A} \to \mathbb{C}$ is assigned. Conversely, to each compact Hausdorff space X, the algebra of continuous functions $f : X \to \mathbb{C}$ is assigned.

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Big aim: generalise Gel'fand-Naimark duality to noncommutative operator algebras, provide spatial counterparts to algebraic constructions.

Introduction (3)

In this talk, I will sketch how some ideas from

- noncommutative operator algebras,
- topos theory,
- geometric model theory,
- and quantum physics

may help to get closer to a solution. Many open questions remain.

The topos approach and Jordan and Lie structures

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More generally, one can think of **UnitC***, the category of unital C^*-algebras and unital $*$-homomorphisms.

The topos approach to quantum theory

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Question: Is $\underline{\Sigma}$ anything like the spectrum of $\mathcal{N}$?

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Here, we consider $\mathcal{N}$ as a Jordan algebra, replacing the noncommutative product with the commutative, but nonassociative symmetrised product

$$\forall \hat{A}, \hat{B} \in \mathcal{N} : \hat{A} \circ \hat{B} := \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}).$$

Automorphisms

Let $\phi:\mathcal{N}\rightarrow\mathcal{N}$ be an ultraweakly continuous Jordan automorphism. This induces

$$\begin{aligned}\tilde{\phi}:\mathcal{V}(\mathcal{N}) &\longrightarrow \mathcal{V}(\mathcal{N})\\ V &\longmapsto \phi(V),\end{aligned}$$

which gives a geometric automorphism $\Phi:\mathbf{Set}^{\mathcal{V}(\mathcal{N})^{\mathrm{op}}}\rightarrow\mathbf{Set}^{\mathcal{V}(\mathcal{N})^{\mathrm{op}}}$. One can use the inverse image part to pull back $\underline{\Sigma}$,

$$\forall V\in\mathcal{V}(\mathcal{N}):(\Phi^{*}(\underline{\Sigma}))_{V}=\underline{\Sigma}_{\tilde{\phi}(V)}.$$

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For each $V \in \mathcal{V}(\mathcal{N})$, we have an isomorphism $\phi|_V : V \to \phi(V)$, such that by Gel'fand duality we get an isomorphism

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The $\mathcal{G}_V$ are the components of a natural transformation $\mathcal{G} : \Phi^*(\underline{\Sigma}) \to \underline{\Sigma}$, so we get an invertible map (automorphism)

$$\mathcal{G} \circ \Phi^* : \underline{\Sigma} \longrightarrow \underline{\Sigma}.$$

Good automorphisms

Observation: In order to reconstruct $\mathcal{N}$, we need not just the symmetrised product (Jordan structure), but also the antisymmetrised product (Lie structure). This 'decomposition' of the noncommutative product also has a good physical motivation.

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The Lie structure is closely related to the unitary group $\mathcal{U}(\mathcal{N})$ acting on $\mathcal{N}$: by Stone's theorem, $\hat{U}_t = e^{it\hat{A}}$ for $\hat{A} \in \mathcal{N}_{\mathrm{sa}}$, and we have

$$\forall \hat{B} \in \mathcal{N}_{\mathrm{sa}} : \frac{d}{dt}(\hat{U}_t \hat{B} \hat{U}_{-t})|_{t=0} = i[\hat{A}, \hat{B}].$$

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Every unitary operator gives a Jordan automorphism $\phi_{\hat{U}} : \mathcal{N} \to \mathcal{N}$, $\hat{A} \mapsto \hat{U}\hat{A}\hat{U}^*$, and hence an automorphism $\mathcal{G} \circ \Phi_{\hat{U}} : \underline{\Sigma} \to \underline{\Sigma}$.

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Aim: Identify these 'good' automorphisms among all automorphisms of $\underline{\Sigma}$. This will help to reconstruct $\mathcal{U}(\mathcal{N})$ and hence the noncommutative von Neumann algebra $\mathcal{N}$.

Morphisms between different algebras

Up to now, we considered only (Jordan) automorphisms of von Neumann algebras. More generally, an ultraweakly continuous unital Jordan morphism

$$\phi : \mathcal{N}_1 \longrightarrow \mathcal{N}_2$$

preserves commutativity and hence induces a geometric morphism $\Phi : \mathbf{Set}^{\mathcal{V}(\mathcal{N}_1)^{\mathrm{op}}} \to \mathbf{Set}^{\mathcal{V}(\mathcal{N}_2)^{\mathrm{op}}}$ and a morphism

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The open task is to identify which of the morphisms $\mathcal{G} \circ \Phi : \underline{\Sigma}_{\mathcal{N}_2} \to \underline{\Sigma}_{\mathcal{N}_1}$ come from (ultraweakly continuous) $*$-homomorphisms $\phi : \mathcal{N}_1 \to \mathcal{N}_2$, i.e., the arrows in **vNA**.

Zariski geometries

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But Lie algebra aspects are not built in yet.

Zariski geometries (2)

There are more connections: one can associate a presheaf (over commutative subalgebras) of Zariski structures with a NC $*$-algebra, see B. Zilber, *Finitary presheaf associated with a non-commutative algebra*, preprint, available from

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Instead of identifying good automorphisms, here a presheaf with richer structure is defined which incorporates Lie algebra aspects.

For details, please see Zilber's website!

Covariant and contravariant

The covariant topos

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By constructive Gel'fand duality (B. Banaschewski/C. Mulvey), the internal algebra has a Gel'fand spectrum $\overline{\Sigma}$ in the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{N})}$.

Combining the two variants?

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Question (suggested by J. Funk): Is there a duality between the pair

$$(\mathbf{Set}^{\mathcal{V}(\mathcal{N})}, \overline{\mathcal{N}}),\ (\mathbf{Set}^{\mathcal{V}(\mathcal{N})^{\mathrm{op}}}, \underline{\Sigma})$$

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If so, this would at least cover the partial (commutative) algebra aspect of $\mathcal{N}$, and presumably also the Jordan algebra aspect.

Thanks for listening!