

# On the Isbell conjugation adjunction for monad-quantale-enriched categories

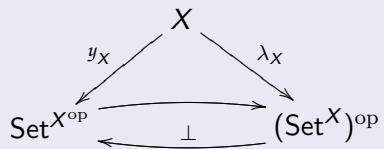
Dirk Hofmann

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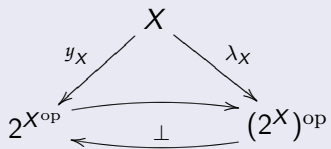
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[dirk@ua.pt](mailto:dirk@ua.pt)

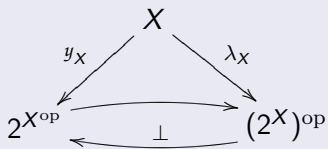
$$\text{hom} : X^{\text{op}} \times X \rightarrow \text{Set}$$



$$\leq: X^{\text{op}} \times X \rightarrow 2$$

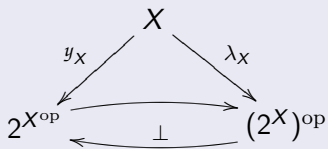


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- Both sides define lax idempotent monads on Ord.
- algebra=(co)complete ordered set.
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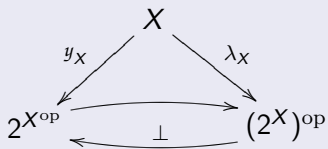
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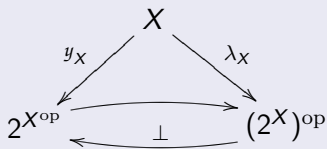


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Now we consider:  $a : (TX)^{\text{op}} \otimes X \rightarrow \mathbb{V}$  with  $\begin{cases} 1_X \leq a \cdot e_X, \\ a \cdot T a \leq a \cdot m_X \end{cases}$

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 $(X, a_0 : X \multimap X) \mapsto (TX, Ta_0 : TX \multimap TX)$ .



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- For  $f : X \rightarrow Y$  and  $(X, a)$ ,  $(Y, b)$  representable:

$$f \text{ is } (\mathbb{T}, \mathbf{V})\text{-functor} \iff \begin{cases} f \text{ is } \mathbf{V}\text{-functor,} \\ f(\alpha(\mathfrak{x})) \geq \beta(Uf(\mathfrak{x})). \end{cases}$$

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- We have

$$\begin{array}{ccc} & X & \\ y_X \swarrow & & \searrow \lambda_X \\ V(TX)^{\text{op}} & \xrightarrow{(-)^+} & (V^X)^{\text{op}} \\ \perp & & \\ \longleftarrow & & \longrightarrow \\ & (-)^- & \end{array}$$

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- For  $X$  repres.:  $X$  cocomplete  $\iff X$  complete.
- However:  $[0, \infty]^{\text{op}}$  (in  $(\mathbb{U}, [0, \infty])\text{-Cat}$ ) is totally complete but not totally cocomplete.

Consider  $f : X \rightarrow Y$  with  $(X, a)$ ,  $(Y, b)$  representable. TFAE:

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- Ⓜ  $f$  is weakly open, i.e.  $f^\circ \cdot b \leq a \cdot Tf^\circ \cdot Tb_0$ .

In Top:

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 x & \overset{\dots\dots\dots}{\rightarrow} & \eta' \leq \eta \\
 \vdots & & \downarrow \\
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Hence, for  $f : X \rightarrow Y$   $(\mathbb{T}, \mathbb{V})$ -functor with  $X, Y$  totally complete:

$$f \text{ is right adjoint in } (\mathbb{T}, \mathbb{V})\text{-Cat} \iff \begin{cases} f \text{ preserves infima} \\ \text{and is "weakly open"}. \end{cases}$$

Let  $X = (X, a)$ ,  $Y = (Y, b)$  be repr.,  $a = a_0 \cdot \alpha$ ,  $b = b_0 \cdot \alpha$ .

$$\varphi : X \rightarrow QY \text{ in } V\text{-Cat}^{\mathbb{T}} = \left\{ \begin{array}{l} \text{V-module } \varphi : X \dashrightarrow Y \text{ where} \\ \begin{array}{ccc} TX & \xrightarrow{T\varphi} & TY \\ \alpha_* \circ \downarrow & & \downarrow \circ \beta_* \\ X & \xrightarrow{\varphi} & Y \end{array} \end{array} \right.$$

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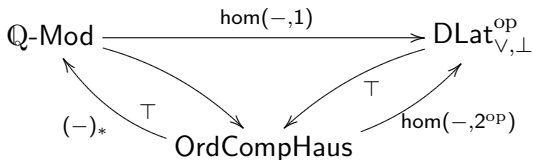
Hence  $(V\text{-Cat}^{\mathbb{T}})_{\mathbb{Q}} \simeq \mathbb{Q}\text{-Mod}$ .

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For Top:



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For Top:

$$\begin{array}{ccc} \mathbb{Q}\text{-Mod} & \xrightarrow{\text{hom}(-,1)} & \text{DLat}_{V,\perp}^{\text{op}} \\ & \searrow & \swarrow \\ & \text{OrdCompHaus} & \end{array}$$

$(-)_*$   $\xrightarrow{\mathbb{T}}$   $\xrightarrow{\text{hom}(-,2^{\text{op}})}$

- $\text{hom}(-,1)$  is induced by a monad morphism  $\delta$ .

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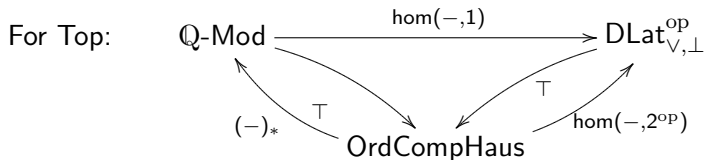
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- $\text{Priest}_{\mathbb{Q}} \simeq \text{DLat}_{V, \perp}^{\text{op}}$ . (Hence:  $\text{Stone}_V \simeq \text{Bool}_{V, \perp}^{\text{op}}$ )