On the Isbell conjugation adjunction for monad-quantale-enriched categories

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$$\mathrm{hom} : X^{\mathrm{op}} \times X \to \mathrm{Set}$$

$$\begin{tikzcd}
 & X \arrow[dl, "y_X"'] \arrow[dr, "\lambda_X"] & \\
\mathrm{Set}^{X^{\mathrm{op}}} \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & (\mathrm{Set}^{X})^{\mathrm{op}} \arrow[ll, bend left]
\end{tikzcd}$$

$$\leq : X^{\mathrm{op}} \times X \to 2$$

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- Both sides define lax idempotent monads on Ord.
- algebra=(co)complete ordered set.
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Now we consider: $a : (TX)^{\mathrm{op}} \otimes X \to \mathsf{V}$ with $\begin{cases} 1_X \leq a \cdot e_X, \\ a \cdot Ta \leq a \cdot m_X \end{cases}$

- $\mathbb{T}$ extends to a monad on V-Cat;
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$$\begin{tikzcd} \text{V-Cat}^{\mathbb{T}} \arrow[rr, bend left, "K"] \arrow[rr, phantom, "\top"] & & (\mathbb{T}, \text{V})\text{-Cat}, \arrow[ll, bend left, "M"] \end{tikzcd} \begin{cases} K(X, a_0, \alpha) = (X, a_0 \cdot \alpha : TV \relbar\joinrel\mapstochar\joinrel\rightarrow V), \\ M(X, a) = (TX, Ta \cdot m_X, m_X) \end{cases}$$

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- For $X = (X, a)$ representable: $X^{\mathrm{op}} = (X, a_0^{\circ} \cdot \alpha)$.

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- $f : X \to Y$ left adjoint, (X, a), (Y, b) representable $\Rightarrow$ f homom.
- For $f : X \to Y$ and (X, a), (Y, b) representable:

$$f \text{ is } (\mathbb{T}, \mathsf{V})\text{-functor} \iff \begin{cases} f \text{ is } \mathsf{V}\text{-functor}, \\ f(\alpha(\mathfrak{x})) \geq \beta(Uf(\mathfrak{x})). \end{cases}$$

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- However: $[0, \infty]^{\mathrm{op}}$ (in $(\mathbb{U}, [0, \infty])$-Cat) is totally complete but not totally cocomplete.

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3. f is weakly open, i.e. $f^\circ \cdot b \leq a \cdot Tf^\circ \cdot Tb_0$.

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Hence, for $f : X \to Y$ $(\mathbb{T}, \mathsf{V})$-functor with X, Y totally complete:

$$f \text{ is right adjoint in } (\mathbb{T}, \mathsf{V})\text{-Cat} \iff \begin{cases} f \text{ preserves infima} \\ \text{and is “weakly open”.} \end{cases}$$

Let $X = (X, a)$, $Y = (Y, b)$ be repr., $a = a_0 \cdot \alpha$, $b = b_0 \cdot \alpha$.

$$\varphi : X \to QY \text{ in V-Cat}^{\mathbb{T}} \quad = \quad \left\{ \begin{array}{l} \text{V-module } \varphi : X \mathrel{\circ\!\!\!\rightarrow} Y \text{ where} \\ \begin{tikzcd} TX \arrow[r, "T\varphi", "\circ"{marking}] \arrow[d, "\alpha_*"', "\circ"{marking}] & TY \arrow[d, "\beta_*", "\circ"{marking}] \\ X \arrow[r, "\varphi"', "\circ"{marking}] & Y \end{tikzcd} \end{array} \right.$$

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- $\mathsf{Priest}_{\mathbb{Q}} \simeq \mathsf{DLat}^{\mathrm{op}}_{\vee,\perp}$. (Hence: $\mathsf{Stone}_{\mathbb{V}} \simeq \mathsf{Bool}^{\mathrm{op}}_{\vee,\perp}$)