The Span Construction on Bicategories

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Span as a profunctor

It is easy to compose a span with a morphism on the left:

$$A \xleftarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

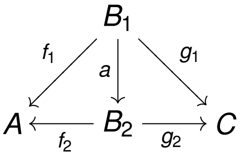
or a backwards morphism on the right:

$$X \xleftarrow{x} A \xleftarrow{f} B \xrightarrow{g} C$$

This makes $\mathcal{S}pan(\mathcal{C})$ into a profunctor $\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathfrak{s}} C$.

Two-cells Between Spans

There is a notion of 2-cell between spans, given by commutative diagrams of the form



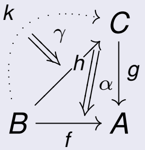
Therefore, $\mathcal{S}pan(\mathcal{C})$ is usually studied as a bicategory. These 2-cells make $\mathfrak{s}$ into a $\mathcal{C}at$-valued profunctor.

Liftings and Profunctors

Definition

In a bicategory $\mathcal{B}$, for objects A, B, C of $\mathcal{B}$,

A Lifting of a morphism $A \xrightarrow{f} B$ along a morphism $C \xrightarrow{g} B$ is a morphism $B \xrightarrow{h} C$ and a 2-cell $gh \xRightarrow{\alpha} f$, such that for any other morphism $B \xrightarrow{k} C$ and 2-cell $gk \xRightarrow{\beta} f$, there is a unique 2-cell $k \xRightarrow{\gamma} h$ such that $\beta = \alpha \bullet (g\gamma)$

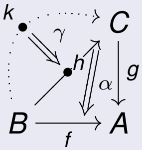


Liftings and Profunctors

Definition

For a profunctor $\mathcal{B} \overset{P}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} \mathcal{C}$, objects A of $\mathcal{B}$, and B, C of $\mathcal{C}$,

A Lifting of an element $A \overset{f}{-\!\!\bullet\!\!\rightarrow} B$ along a morphism $C \xrightarrow{g} B$ is an element $B \overset{h}{-\!\!\bullet\!\!\rightarrow} C$ and a 2-cell $gh \overset{\alpha}{\Longrightarrow} f$, such that for any other element $B \overset{k}{-\!\!\bullet\!\!\rightarrow} C$ and 2-cell $gk \overset{\beta}{\Longrightarrow} f$, there is a unique 2-cell $k \overset{\gamma}{\Longrightarrow} h$ such that $\beta = \alpha \bullet (g\gamma)$

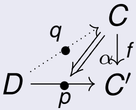


Lifting-Absolute Profunctors

Definition

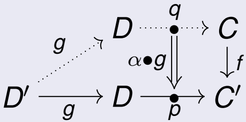
A $\mathcal{C}at$-valued profunctor $\mathcal{D} \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;} \mathcal{C}$ is lifting-absolute if

For any lifting



and any morphism $D' \xrightarrow{g} D$ in $\mathcal{D}$,

the composite



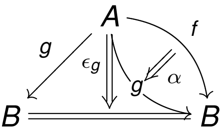
is also a lifting.

Proposition

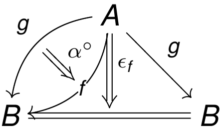
For any profunctor $\mathcal{D} \overset{Q}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;} \mathcal{C}^{\mathrm{op}}$, the composite $\mathcal{D} \overset{Q}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;} \mathcal{C}^{\mathrm{op}} \overset{\mathfrak{s}}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;} \mathcal{C}$ is lifting-absolute.

Morphisms Between Spans on a Bicategory

If we demand that $B \xleftarrow{f} A$ be a lifting of the identity along $A \xrightarrow{f} B$, then for a 2-cell $A \underset{g}{\overset{f}{\Downarrow \alpha}} B$, we have a pasting diagram:

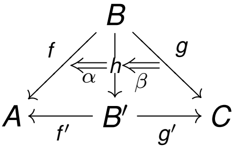


which must factor uniquely as

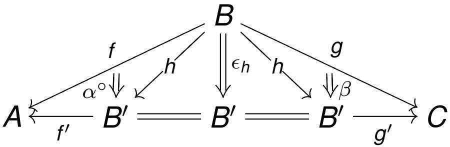


Morphisms Between Spans on a Bicategory

From these morphisms, we can construct morphisms of the form



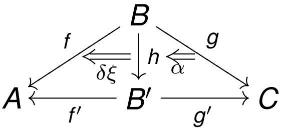
As the composites



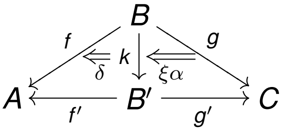
Subject to an equivalence relation.

Invertible 2-cells

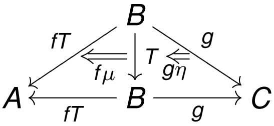
The equivalence relation imposed says that the morphisms



and



are equal. We take the transitive closure of this relation. This means that there can be multiple representations of the identity morphism. For example, if (T, η, μ) is a monad on B, then



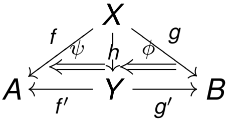
Is a representation of the identity.

Injunctions

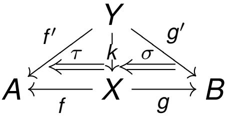
- Representations of the identity can be arbitrarily complicated.
- This means that deciding whether a morphism is invertible is difficult.
- We therefore restrict attention to cases where the structure isomorphisms have a particularly simple form.
- We will call these invertible morphisms injunctions.

Definition

An injunction consists of a pair of inverse morphisms of spans θ and θ^{-1} with representations



and



respectively, and 2-cells $1_X \overset{\eta}{\Longrightarrow} kh$ and $hk \overset{\epsilon}{\Longrightarrow} 1_Y$ such that:

$$\sigma h \circ \phi = g\eta \qquad (1) \qquad\qquad g'\epsilon \circ \phi k \circ \sigma = 1_{g'} \qquad (3)$$

$$\psi \circ \tau h \circ f\eta = 1_f \qquad (2) \qquad\qquad \tau \circ \psi k = f'\epsilon \qquad (4)$$

In this case, we refer to θ^{-1} as the injoint of θ, and vice versa.

Composable pairs of Spans

Composable pairs of spans also form a profunctor, given as the composite

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathfrak{s}} \mathcal{C} \xrightarrow{\hat{\mathfrak{c}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathfrak{s}} \mathcal{C}$$

where $\hat{\mathfrak{c}}$ is the cospan profunctor, with strict identities added. Composition of spans is given by a natural transformation of profunctors $\mathfrak{s}\hat{\mathfrak{c}}\mathfrak{s} \xrightarrow{\alpha} \mathfrak{s}$.

Canonical Squares

Often, general compositions of spans can be derived from compositions of the form:

$$A = A \xrightarrow{f} B \xleftarrow{g} C = C$$

We will call compositions of this form canonical squares.

$$\begin{array}{ccc} P & \xrightarrow{p} & C \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

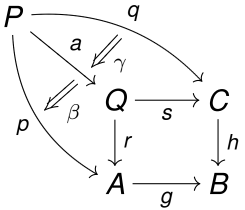
Examples

- Pseudopullbacks
- Comma Squares

Horizontal Composition

If
$$\begin{array}{ccc} P & \xrightarrow{q} & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow[f]{} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} Q & \xrightarrow{s} & C \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow[g]{} & B \end{array}$$

are canonical squares, then for any 2-cell $f \overset{\alpha}{\Longrightarrow} g$, there is a morphism $P \xrightarrow{a} Q$, and a pair of 2-cells β and γ:

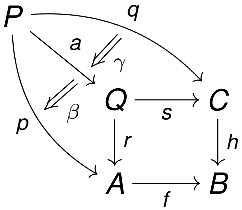


subject to some coherence and functoriality conditions.

Horizontal Composition

If
$$\begin{array}{ccc} P & \xrightarrow{q} & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$
and
$$\begin{array}{ccc} Q & \xrightarrow{s} & C \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow[f]{} & B \end{array}$$

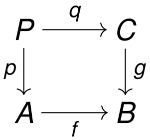
are canonical squares, then for any 2-cell $h \overset{\alpha}{\Longrightarrow} g$, there is a morphism $P \xrightarrow{a} Q$, and a pair of 2-cells β and γ:



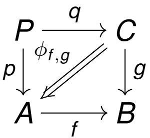
subject to some coherence and functoriality conditions.

A Particular Composite

Any canonical square



must contain a chosen 2-cell, $\phi_{f,g}$.



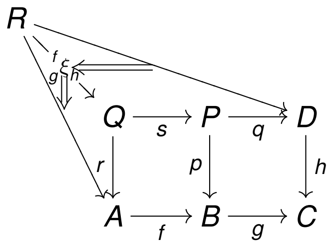
subject to the obvious composition rules.

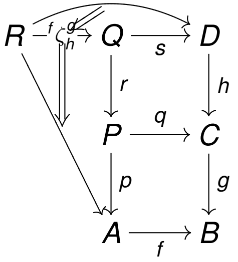
Structure Isomorphisms

The structure isomorphisms look fairly standard when expressed in terms of α. Namely, there is an isomorphism

$$\alpha(\alpha\mathfrak{c}\mathfrak{s}) \xrightarrow{\theta} \alpha(\mathfrak{s}\mathfrak{c}\alpha)$$

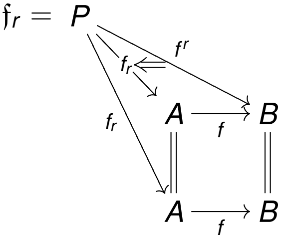
Which we will demand is an injunction. This corresponds to a pair of injunctions between canonical squares:

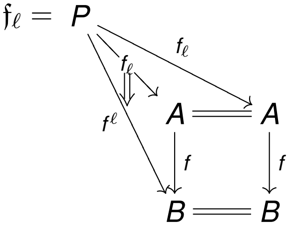
$\Xi =$


$Z =$


Unit Structure Isomorphisms

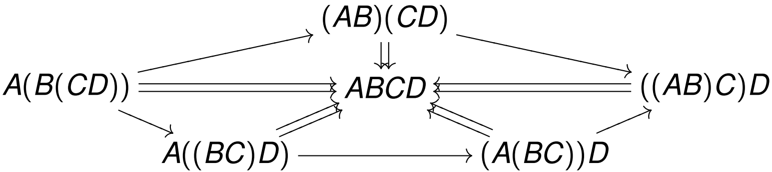
We also have structure isomorphisms corresponding to the unit laws:





Coherence Conditions

The associator isomorphism gives us a collection of isomorphisms:



To prove coherence of the outside pentagon, we need to show that the pairs of isomorphisms from each outside vertex to the centre are equal.

Coherence Conditions

Showing this boils down to demanding 3 sorts of coherence conditions on the isomorphisms ζ and ξ, corresponding to the diagrams:

- $$\begin{array}{ccccccc} R & \longrightarrow & Q & \longrightarrow & P & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \end{array}$$

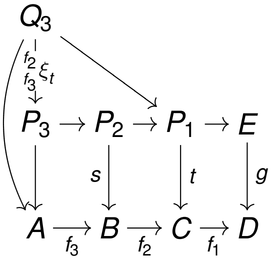
- $$\begin{array}{ccccc} S & \longrightarrow & R & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ Q & \longrightarrow & P & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C \end{array}$$

- $$\begin{array}{ccc} R & \longrightarrow & E \\ \downarrow & & \downarrow \\ Q & \longrightarrow & D \\ \downarrow & & \downarrow \\ P & \longrightarrow & C \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

Examples of Coherence Conditions

$$\begin{array}{ccccccc}
P_3 & \rightarrow & P_2 & \rightarrow & P_1 & \rightarrow & E \\
\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle g} \\
A & \xrightarrow[f_3]{} & B & \xrightarrow[f_2]{} & C & \xrightarrow[f_1]{} & D
\end{array}$$

Examples of Coherence Conditions



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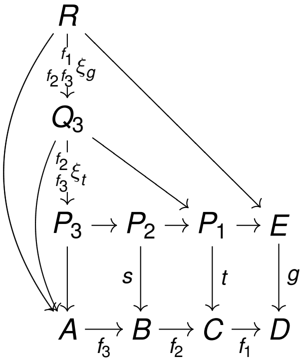

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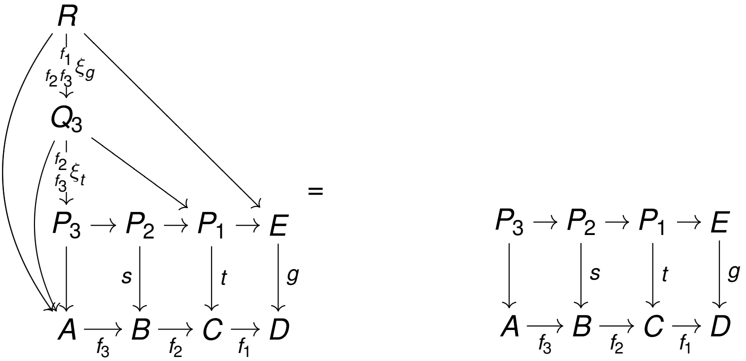

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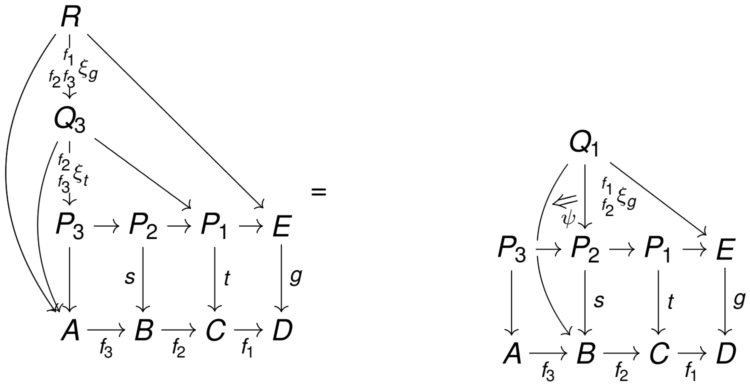

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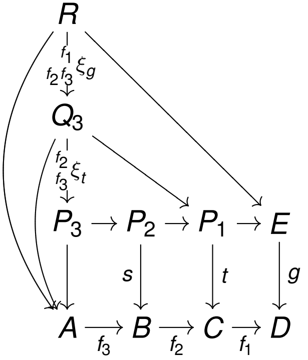

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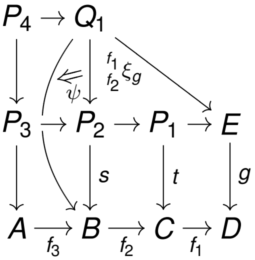

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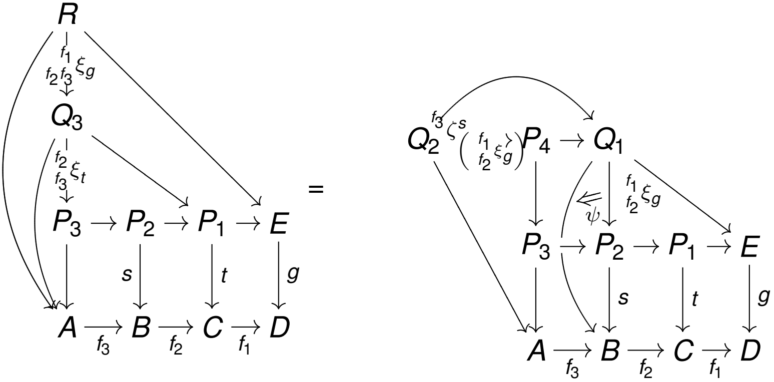

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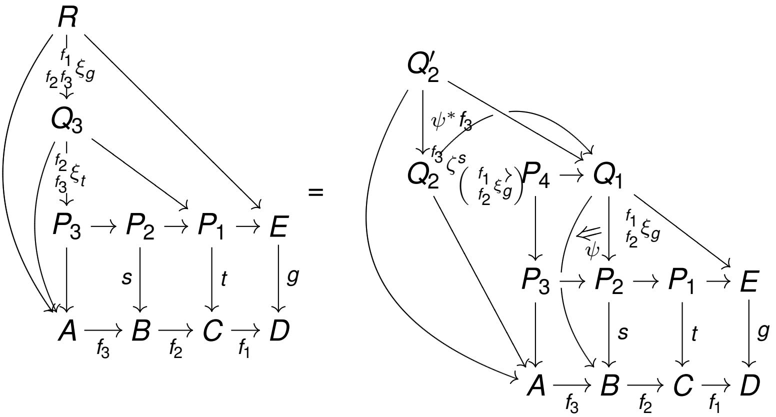

Examples of Coherence Conditions