Notions of Möbius inversion

Tom Leinster

Glasgow/EPSRC

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That is, if **A** and $\mathbf{A}'$ have the same underlying graph then $\chi(|\mathbf{A}|) = \chi(|\mathbf{A}'|)$.

Plan

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1. A simplified history of Möbius inversion

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Overview

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Number-theoretic Möbius inversion
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Coarse M. inversion for categories
(Haigh 1980;
Leinster 2008)

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Important in number theory, e.g.

$$1 \Big/ \sum_n \frac{1}{n^s} = \sum_n \frac{\mu(n)}{n^s}.$$

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E.g.: $(A, \leq) = (\mathbb{Z}^+, |)$: then $\mu(a, b) = \mu_{\text{classical}}(b/a)$.

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Zeta function	$\zeta(f) \equiv 1$	$\zeta(a, b) = \lvert\mathrm{Hom}(a, b)\rvert$

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2. Fine vs. coarse Möbius inversion

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Number-theoretic Möbius inversion
(Möbius 1832)

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- Throwing away the composition of a category is extravagant...
  but it's surprising how much remains.

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