# Pseudo-commutativity of lax-idempotent 2-monads

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# Plan

1. Pseudo-closed categories.
2. Pseudo-commutativities.
3. Lax-idempotent 2-monads.
4. Lax-idempotent 2-monads are pseudo-commutative.
5. Few comments on finite colimits.

# Motivation

Given $A, B$ $\mathbb{V}$-categories with finite colimits, $\exists$ a *tensor product* that classifies $A \otimes B \to C$ rex in each variable.

$$A \otimes B \to A \boxtimes B$$

induces

$$\mathbf{REX}(A \boxtimes B, C) \simeq \mathbf{REX}(A, B; C)$$

$A \boxtimes B$ is defined only up to *equivalence.*

- How to get $\boxtimes$ with better properties? (relate to Deligne's tensor product of abelian categories).
- Need: algebraic formulation of commutation of colimits with colimits.

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# Pseudo-closed 2-categories

$$\textbf{SymmMonCat} = \begin{cases} \text{Strict symmetric monoidal categories} \\ \text{braided monoidal functors} \\ \text{and monoidal transformations} \end{cases}$$

$$= S\text{-}\textbf{Alg} \text{ for a 2-monad } S : \textbf{Cat} \to \textbf{Cat}$$

## Remark

- $S$-**Alg** has internal homs: braided monoidal functors $A \to B$ form a symm. strict monoidal category $[\![A, B]\!]$.
- This is not a closed structure: $[\![S(1), B]\!] \simeq B$ *not* an isomorphism.

# Pseudo-closed 2-categories

## Definition (Eilenberg-Kelly)

A *closed* 2-category is $\mathcal{K}$ with

- $K : \mathcal{K} \to \mathbf{Cat}$.
- Internal homs $[\![-,-]\!] : \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$ s.t. $K[\![-,-]\!] = \mathcal{K}(-,-)$.
- An object $I \in \mathcal{K}$.
- Transformations
  - $id : I \to [\![A,A]\!]$.
  - $comp : [\![B,C]\!] \to [\![[\![A,B]\!],[\![A,C]\!]]\!]$.
  - Isomorphisms $[\![I,B]\!] \cong B$.

+ axioms.

# Pseudo-closed 2-categories

## Definition (Hyland-Power)

A *pseudo-closed* 2-category is $\mathcal{K}$ with

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  - $id : I \to [\![A,A]\!]$.
  - $comp : [\![B,C]\!] \to [\![[\![A,B]\!],[\![A,C]\!]]\!]$.
  - Retract equivalences $i_B \dashv e_B : [\![I,B]\!] \simeq B$.

+ axioms (associativity of composition, compatibility of the retract equivalences with identities and composition, etc.).

# Pseudo-commutativities

## Definition (Kock)

A $\mathbb{V}$-monad $T : \mathbb{V} \to \mathbb{V}$ is *commutative* when

$$\begin{array}{ccccc} TX \otimes TY & \xrightarrow{t} & T(TX \otimes Y) & \xrightarrow{Tt'} & T^2(X \otimes Y) \\ \downarrow{\scriptstyle t'} & & & & \downarrow{\scriptstyle \mu} \\ T(X \otimes TY) & \xrightarrow{Tt} & T^2(X \otimes Y) & \xrightarrow{\mu} & T(X \otimes Y) \end{array}$$

where $t : X \otimes TY \to T(X \otimes Y)$ is the strength corresponding to the enrichment $T : [X, Y] \to [TX, TY]$.

## Example

$S : \mathbf{Cat} \to \mathbf{Cat}$ free symm. strict monoidal category 2-monad.

$$\begin{array}{ccc} ((x_1, x_2), (y_1, y_2)) & \longmapsto & ((x_1, y_1), (x_2, y_1), (x_1, y_1), (x_2, y_2)) \\ & \searrow & \uparrow \cong \\ & & ((x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)) \end{array}$$

# Pseudo-commutativities

## Definition (Hyland-Power)

A *pseudo-commutativity* on a 2-monad $T : \mathbf{Cat} \to \mathbf{Cat}$ is an invertible modification satisfying some axioms.

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TX \times TY & \xrightarrow{t} & T(TX \times Y) & \xrightarrow{Tt'} & T^2(X \times Y) \\
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# Pseudo-commutativities

**Theorem (Hyland-Power)**

*If $T$ is pseudo-commutative then $T$-**Alg** is pseudo-closed.*

*Under mild assumptions there is a corresponding tensor product*

$$T\text{-}\mathbf{Alg}(A \boxtimes B, C) \simeq T\text{-}\mathbf{Alg}(A, [\![B, C]\!])$$

$A \boxtimes B$ classifies "multilinear maps" $A \otimes B \to C$.

$$- \boxtimes B = \left( T\text{-}\mathbf{Alg} \underset{J}{\overset{(-)'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} T\text{-}\mathbf{Alg}_s \underset{[\![B,-]\!]}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}} T\text{-}\mathbf{Alg}_s \xrightarrow{J} T\text{-}\mathbf{Alg} \right)$$

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# A characterisation of pseudo-commutativity

Recall: $T$ is commutative iff $\sigma_{X,B}$ is an algebra morphism, for all $X \in \mathbb{V}, B \in \mathbb{V}^T$.

$$\sigma_{X,B} : [X,B] \xrightarrow{T} [TX,TB] \xrightarrow{[1,b]} [TX,B]$$

**Theorem**

*There is a bijection between pseudo-commutativities on $T$ and liftings of the pseudonatural*

$$\sigma : [-,U] \Rightarrow [T-,U-] : \mathbf{Cat}^{op} \times T\text{-}\mathbf{Alg} \to \mathbf{Cat}$$

*to a pseudonatural*

$$\sigma : [-,-] \Rightarrow [T-,-] : \mathbf{Cat}^{op} \times T\text{-}\mathbf{Alg} \to T\text{-}\mathbf{Alg}$$

*that is 2-natural on* strict *morphisms + satisfying axioms.*

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*that is 2-natural on* strict *morphisms + satisfying axioms.*

Want to apply this theory to categories with a class of (co)limits.

# Colimits

## Theorem (Lack-Kelly)

*Given a class of $\mathbb{V}$-enriched colimits* (e.g., *finite colimits*) *there is a 2-monad on $\mathbb{V}$-**Cat** whose algebras are $\mathbb{V}$-categories with chosen colimits of that class.*

$$(\text{class of colimits } \Phi) \rightsquigarrow (\text{2-monad } T_\Phi \text{ on } \mathbb{V}\text{-}\mathbf{Cat})$$

and

$$T_\Phi\text{-}\mathbf{Alg} = \begin{cases} \mathbb{V}\text{-categories with chosen } \Phi\text{-colimits} \\ \Phi\text{-cocontinuous } \mathbb{V}\text{-functors} \\ \mathbb{V}\text{-natural transformations} \end{cases}$$

Recall: a $\mathbb{V}$-category $A$ is $\Phi$-cocomplete iff the unit $A \to T_\Phi A$ admits a left adjoint.

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# Lax-idempotent 2-monads

## Definition

A 2-monad $T : \mathcal{K} \to \mathcal{K}$ is *lax-idempotent* (or *K-Z*) when any of the following equivalent conditions hold.

- Any 1-cell in $\mathcal{K}$ between $T$-algebras has a unique lax morphism structure.
- For any $T$-algebra $(A, a)$ there is an adjunction $a \dashv \eta_A : A \to TA$ with counit an identity.
- Many more equivalent conditions.

## Example

$T_\Phi$ for a class of colimits $\Phi$.

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$T_\Phi$ for a class of colimits $\Phi$.

## Theorem

*Any lax-idempotent 2-monad $T$ on* **Cat** *has a unique pseudo-commutativity.*

Proof.

1. $\sigma_{A,B} : [A,B] \xrightarrow{T} [TA,TB] \xrightarrow{[1,b]} [TA,B]$ left adjoint to $[\eta_A, B]$.
2. Left adjoints are pseudomorphisms.
3. Apply characterisation theorem of earlier.

□

**Theorem**

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# Finite colimits

Let $R : \mathbb{V}\text{-}\mathbf{Cat} \to \mathbb{V}\text{-}\mathbf{Cat}$ the monad corresp. to finite colimits.

## Remark

- Pseudo-commutativity of $R$ = "colimits commute with colimits."
- $R\text{-}\mathbf{Alg}(A \boxtimes B, C) \simeq R\text{-}\mathbf{Alg}(A, [\![B, C]\!])$ means that $A \boxtimes B$ classifies functors $A \otimes B \to C$ rex in each variable.

Consequence: this $\boxtimes$ preserves filtered colimits of strict maps in each variable.

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# References

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