(Co-)lax idempotent pseudomonads and Kan pseudomonads

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Manes' exercise on monads

A monad $\mathbb{S}$ on a category **C** is equivalent to:

- A *function* $|S| : |\mathbf{C}| \to |\mathbf{C}|$;
- for every $A \in \mathbf{C}$, an arrow $\eta A : A \to SA$;
- for every morphism $f : B \to SA$ in **C**, an $\mathbb{S}$-extension $f^{\mathbb{S}} : SB \to SA$.

Subject to the axioms:

- for every A in **C**,

$$(\eta A)^{\mathbb{S}} = 1_{SA};$$

- for every $f : B \to SA$ in **C** and $g : C \to SB$, the diagrams

$$\begin{array}{ccc} B & \xrightarrow{\eta B} & SB \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f^{\mathbb{S}}} \\ & & SA, \end{array} \qquad \begin{array}{ccc} SC & \xrightarrow{g^{\mathbb{S}}} & SB \\ & {\scriptstyle (f^{\mathbb{S}}\cdot g)^{\mathbb{S}}}\searrow & \downarrow{\scriptstyle f^{\mathbb{S}}} \\ & & SA \end{array}$$

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The algebras for $\mathbb{S}$

An $\mathbb{S}$-algebra $\mathbb{B} = (B, (-)^{\mathbb{B}})$ consists of:

- An object B in $\mathbf{C}$;
- for every arrow $h : X \to B$ in $\mathbf{C}$, an extension $h^{\mathbb{B}} : SX \to B$;
  Subject to the commutativity of the diagrams
  (with $h : X \to B$ and $y : Y \to SX$):

$$\begin{array}{ccc} X & \xrightarrow{\eta X} & SX \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle h^{\mathbb{B}}} \\ & & B, \end{array} \qquad \begin{array}{ccc} SY & \xrightarrow{y^{\mathbb{S}}} & SX \\ & {\scriptstyle (h^{\mathbb{B}} y)^{\mathbb{B}}}\searrow & \downarrow{\scriptstyle h^{\mathbb{B}}} \\ & & B. \end{array}$$

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A morphism of $\mathbb{S}$-algebras $(B, (-)^{\mathbb{B}})$ to $(A, (-)^{\mathbb{A}})$ is

an arrow $\ell : B \to A$ in $\mathbf{C}$

subject to the commutativity of the diagram

$$\begin{array}{ccc} SX & \xrightarrow{h^{\mathbb{B}}} & B \\ & {\scriptstyle (\ell\cdot h)^{\mathbb{A}}} \searrow & \downarrow{\scriptstyle \ell} \\ & & A. \end{array}$$

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The algebras for $\mathbb{S}$

Theorem. The category of usual algebras for the monad $\mathbb{S}$ is isomorphic to the category of algebras just defined.

Distributive laws

$\mathbb{S} = (S, \eta_S, (-)^{\mathbb{S}})$, $\mathbb{T} = (T, \eta_T, (-)^{\mathbb{T}})$ monads on **C**.

A distributive law of $\mathbb{S}$ over $\mathbb{T}$ can be given as follows:

- For every A in **C** an $\mathbb{S}$-algebra $(TSA, (-)^{\lambda})$

Subject to the axioms

- for every A in **C**, $(T\eta_S A \cdot \eta_T A)^{\lambda} = \eta_T SA$;
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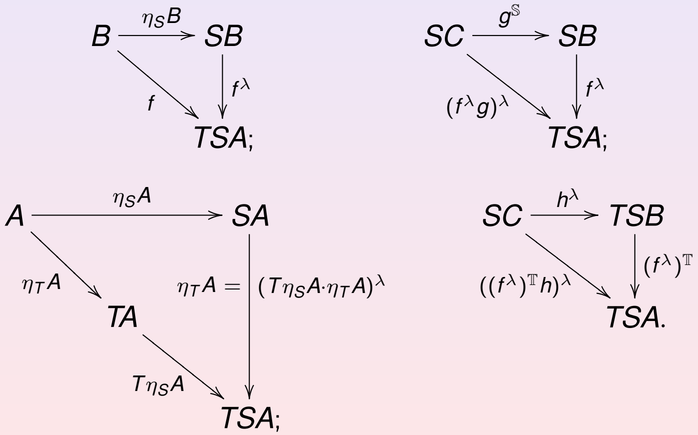
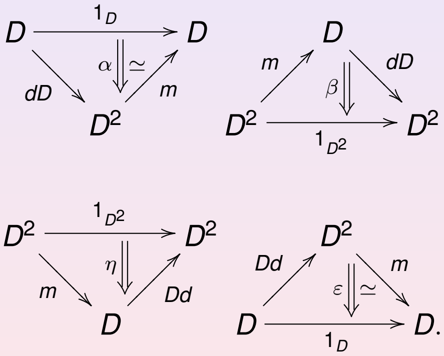
All the diagrams together

$f : B \to TSA,$
$g : C \to SB,$
$h : C \to TSB.$



Colax idempotent pseudomonads

$\mathbb{D} = (D, d, m, \alpha, \beta, \eta, \varepsilon)$
on $\mathcal{K}$, is given by $dD \dashv m \dashv Dd$:



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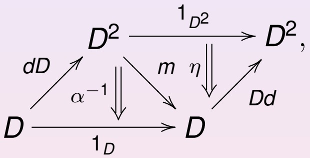
$\delta : dD \to Dd$



Kan pseudomonads

Definition

A **right Kan pseudomonad** $\mathbb{D}$ on $\mathcal{K}$ is given as follows:

i) A *function* $D : \mathrm{Ob}(\mathcal{K}) \to \mathrm{Ob}(\mathcal{K})$.

ii) For every $\mathbf{A} \in \mathcal{K}$, a 1-cell $d\mathbf{A} : \mathbf{A} \to D\mathbf{A}$.

iii) For every 1-cell $F : \mathbf{B} \to D\mathbf{A}$, a right Kan extension of F along $d\mathbf{B}$

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{d\mathbf{B}} & D\mathbf{B} \\ & \searrow^{F} \quad \Swarrow \mathbb{D}_F & \downarrow F^{\mathbb{D}} \\ & & D\mathbf{A} \end{array}$$

with $\mathbb{D}_F$ invertible.

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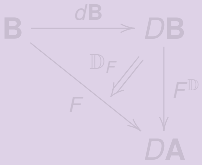

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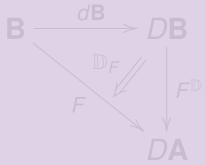

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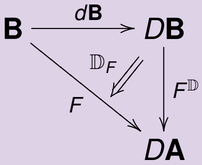

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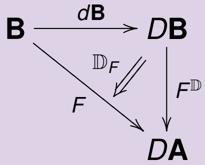

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Subject to the axioms

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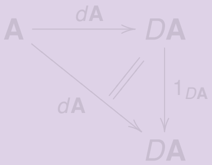

exhibits $1_{D\mathbf{A}}$ as a right Kan extension of $d\mathbf{A}$ along $d\mathbf{A}$.

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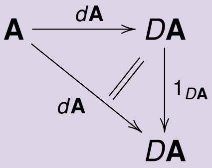

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Kan pseudomonads

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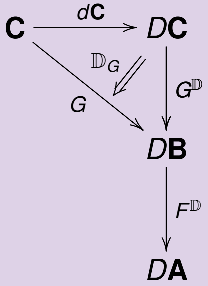
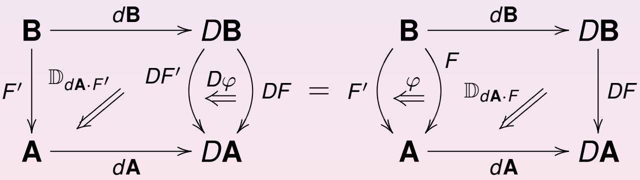
b) For every $G : \mathbf{C} \to D\mathbf{B}$ and $F : \mathbf{B} \to D\mathbf{A}$ the 2-cell



exhibits $F^{\mathbb{D}}G^{\mathbb{D}}$ as a right Kan extension of $F^{\mathbb{D}}G$ along $d\mathbf{C}$.

A right Kan pseudomonad induces a colax idempotent pseudomonad

First we must define a pseudofunctor $D : \mathcal{K} \to \mathcal{K}$.
For $F : \mathbf{B} \to \mathbf{A}$, $DF := (d\mathbf{A} \circ F)^{\mathbb{D}}$, $d_F := \mathbb{D}_{d\mathbf{A} \cdot F}$.
For $\varphi : F \to F' : \mathbf{B} \to \mathbf{A}$, $D\varphi$ is the unique 2-cell such that



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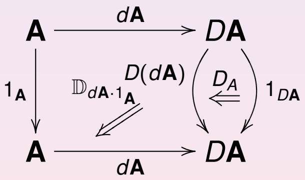
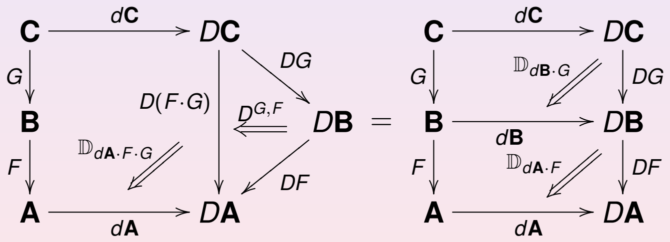
For $\mathbf{A}$ in $\mathcal{K}$,
define $D_A : 1_{D\mathbf{A}} \to D(1_{\mathbf{A}})$
as the unique 2-cell such that



$= 1_{d\mathbf{A}}$.

A right Kan pseudomonad induces a colax idempotent pseudomonad

For $F : \mathbf{B} \to \mathbf{A}$ and $G : \mathbf{C} \to \mathbf{B}$,
define $D^{G,F} : DF \cdot DG \to D(F \cdot G)$
as the unique 2-cell such that



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Define $m : D^2 \to D$ such that for every $\mathbf{A}$,

$$m\mathbf{A} = 1_{D\mathbf{A}}{}^{\mathbb{D}}.$$

For $F : \mathbf{B} \to \mathbf{A}$,
Define $m_F : DF \cdot m\mathbf{B} \to m\mathbf{A} \cdot D^2 f$
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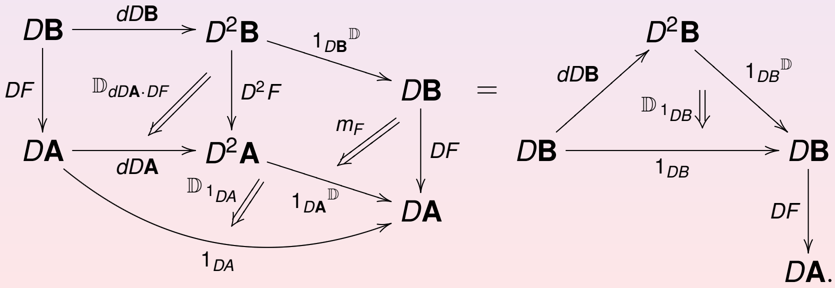
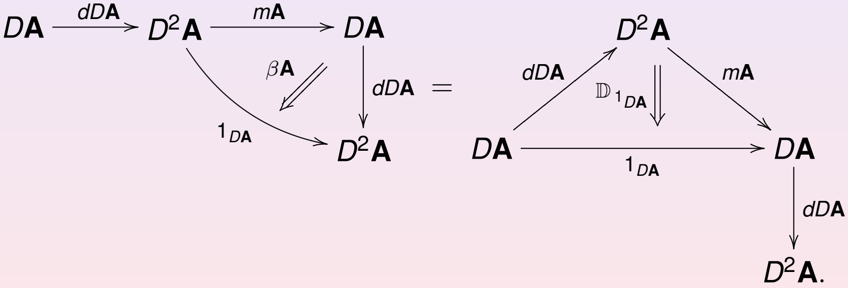


A right Kan pseudomonad induces a colax idempotent pseudomonad

$\alpha\mathbf{A} = {\mathbb{D}_{1_{D\mathbf{A}}}}^{-1}$

$\beta\mathbf{A} : dD\mathbf{A} \cdot m\mathbf{A} \to 1_{D^2\mathbf{A}}$ as the unique 2-cell such that



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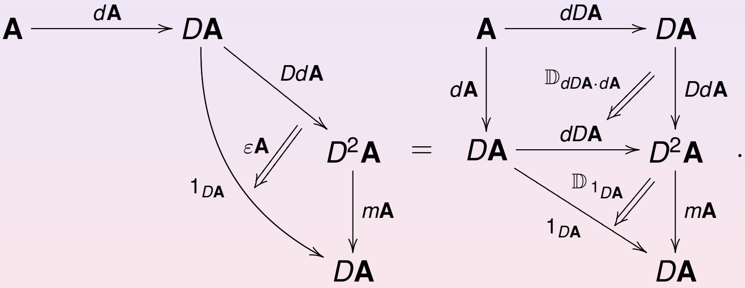
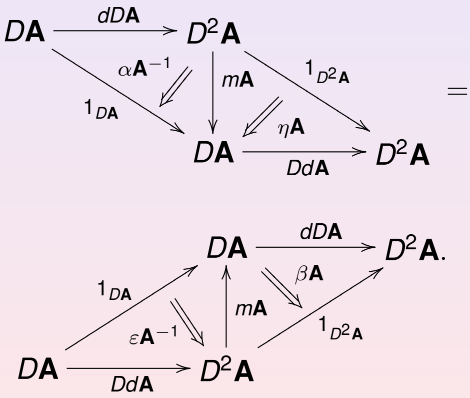
$\varepsilon : m\mathbf{A} \cdot Dd\mathbf{A} \to 1_{D\mathbf{A}}$ as the unique 2-cell such that



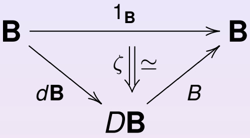
A right Kan pseudomonad induces a colax idempotent pseudomonad

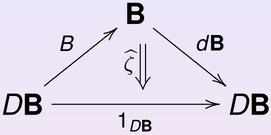
$\eta : 1_{D^2\mathbf{A}} \to Dd\mathbf{A} \cdot m\mathbf{A}$ as the unique 2-cell such that



Algebras for a colax idempotent pseudomonad

Recall that the algebras are adjunctions $\zeta, \widehat{\zeta} : B \dashv d\mathbf{B}$,

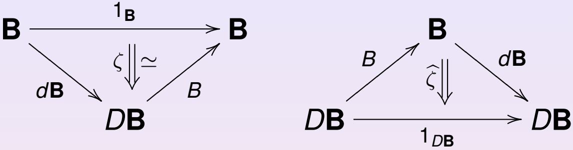




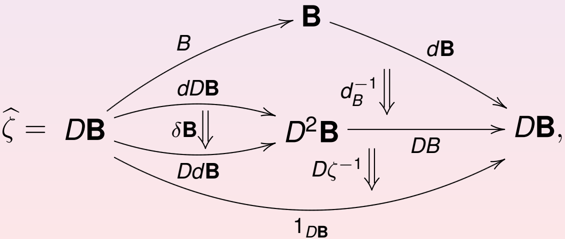
with invertible unit.

Algebras for a colax idempotent pseudomonad

Recall that the algebras are adjunctions $\zeta, \widehat{\zeta} : B \dashv d\mathbf{B}$,

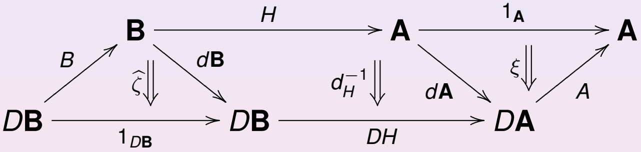


with invertible unit.



Algebras for a colax idempotent pseudomonad

A 1-cell from $\zeta : 1_{\mathbf{B}} \to B \cdot d\mathbf{B}$ to $\xi : 1_{\mathbf{A}} \to A \cdot d\mathbf{A}$ is a 1-cell $H : \mathbf{B} \to \mathbf{A}$ such that the the pasting



is invertible.

Given $H, K : \zeta \to \xi$, a 2-cell in $\mathbb{D}$-Alg is a 2-cell $\tau : H \to K$ in $\mathcal{K}$.

Algebras for a colax idempotent pseudomonad

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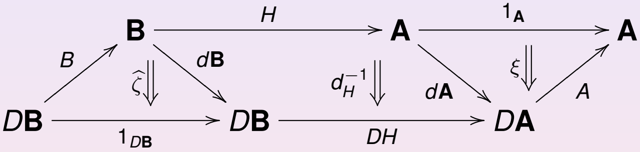

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The algebras for a right Kan pseudomonad

An object $\mathbb{B}$ consists of an object $\mathbf{B}$
together with an assignment, to every $F : \mathbf{C} \to \mathbf{B}$,
of a right Kan extension $F^{\mathbb{B}} : D\mathbf{C} \to \mathbf{B}$ of F along $d\mathbf{C}$

with $\mathbb{B}_F$ invertible,

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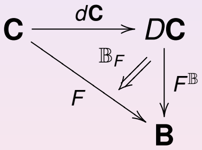

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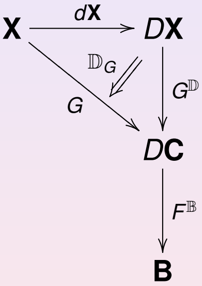
The algebras for a right Kan pseudomonad

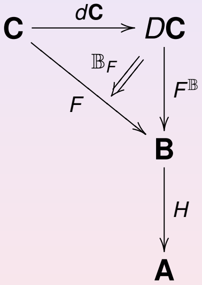
such that for every $G\colon \mathbf{X} \to D\mathbf{C}$,
the diagram



exhibits $F^{\mathbb{B}} \cdot G^{\mathbb{D}}$ as a right Kan extension of $F^{\mathbb{B}} \cdot G$ along $d\mathbf{X}$.

The algebras for a right Kan pseudomonad

A 1-cell $H : \mathbb{B} \to \mathbb{A}$
is a 1-cell $H : \mathbf{B} \to \mathbf{A}$ in $\mathcal{K}$
such that for every $F : \mathbf{C} \to \mathbf{B}$, the diagram



exhibits $F^{\mathbb{B}} \cdot H$ as a right Kan extension of $F \cdot H$ along $d\mathbf{C}$.
A 2-cell $\tau : H \to K : \mathbb{B} \to \mathbb{A}$ is a 2-cell $\tau : H \to K$ in $\mathcal{K}$.

The algebras for a right Kan pseudomonad

Theorem. The 2-category of algebras for a right Kan pseudomonad is biequivalent to the usual 2-category of algebras for the induced colax idempotent pseudomonad.

Transitions

Definition

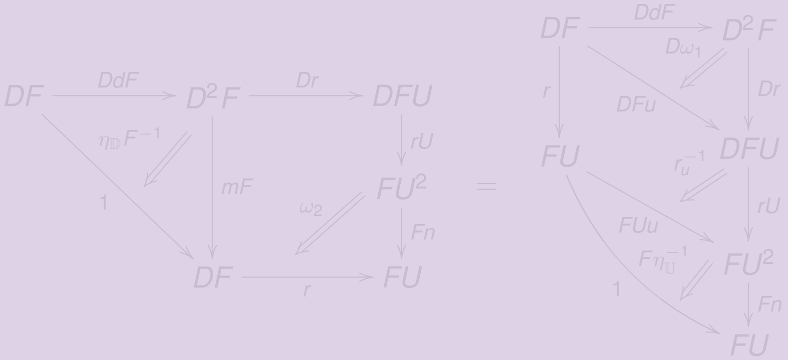
A transition from $\mathbb{U}$ to $\mathbb{D}$ along $F : \mathcal{A} \to \mathcal{B}$ is a strong transformation $r : DF \to FU$ together with invertible modifications

$$
\begin{array}{ccc}
F & \xrightarrow{dF} & DF \\
& \searrow^{Fu} \;\; \overset{\omega_1}{\swarrow\!\!\!\swarrow} & \downarrow r \\
& & FU,
\end{array}
\qquad
\begin{array}{ccccc}
D^2F & \xrightarrow{Dr} & DFU & \xrightarrow{rU} & FU^2 \\
{\scriptstyle mF}\downarrow & & & \overset{\omega_2}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle Fn} \\
DF & & \xrightarrow[r]{\hspace{4em}} & & FU
\end{array}
$$

Transitions

Definition

that satify the following coherence conditions:



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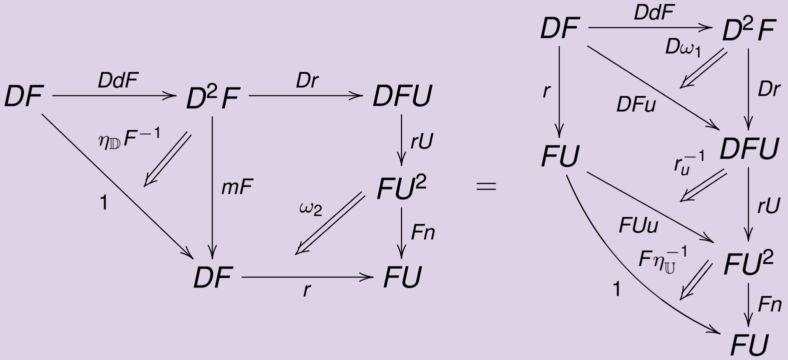

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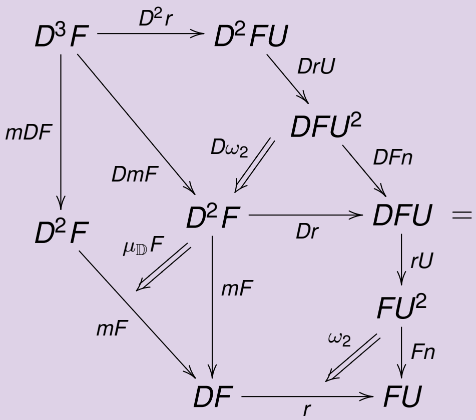

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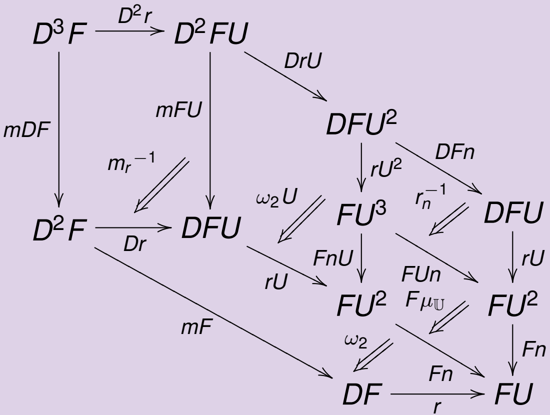

Transitions for right Kan pseudomonads

Theorem

$\mathbb{U}$ and $\mathbb{D}$ right Kan pseudomonads on 2-categories $\mathcal{L}$ and $\mathcal{K}$ respectively.

A transition from $\mathbb{U}$ to $\mathbb{D}$ along a 2-functor $F : \mathcal{L} \to \mathcal{K}$ is given as follows:

for every $\mathbf{A}$ in $\mathcal{L}$, a $\mathbb{D}$-algebra $(FU\mathbf{A}, (\ )^{\lambda})$, such that for every $L : \mathbf{B} \to U\mathbf{A}$ in $\mathcal{L}$,

$$F(L^{\mathbb{U}}) : (FU\mathbf{B}, (\ )^{\lambda}) \to (FU\mathbf{A}, (\ )^{\lambda})$$

is a morphism of $\mathbb{D}$-algebras.

Every transition from $\mathbb{U}$ to $\mathbb{D}$ along F is coherently isomorphic to one that arises in this way.

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Transitions for right Kan pseudomonads

Proof. $r\mathbf{A} = (Fu\mathbf{A})^{\lambda}$

$$\omega_1\mathbf{A} = \begin{array}{ccc} F\mathbf{A} & \xrightarrow{dF\mathbf{A}} & DF\mathbf{A} \\ & \searrow^{Fu\mathbf{A}} \;\; \Swarrow \omega_1\mathbf{A} = \lambda_{Fu\mathbf{A}} & \downarrow (Fu\mathbf{A})^{\lambda} = r\mathbf{A} \\ & & FU\mathbf{A} \end{array}$$

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\begin{array}{ccc}
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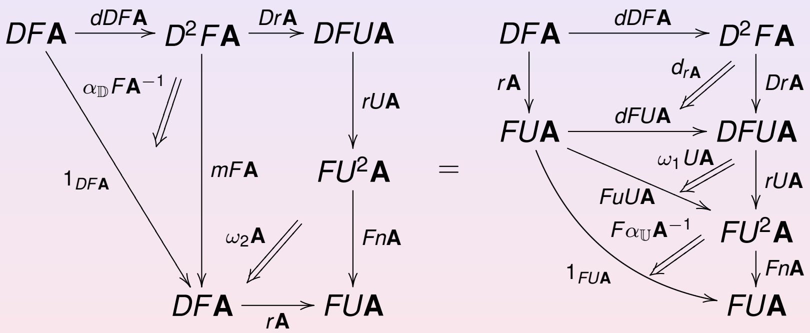
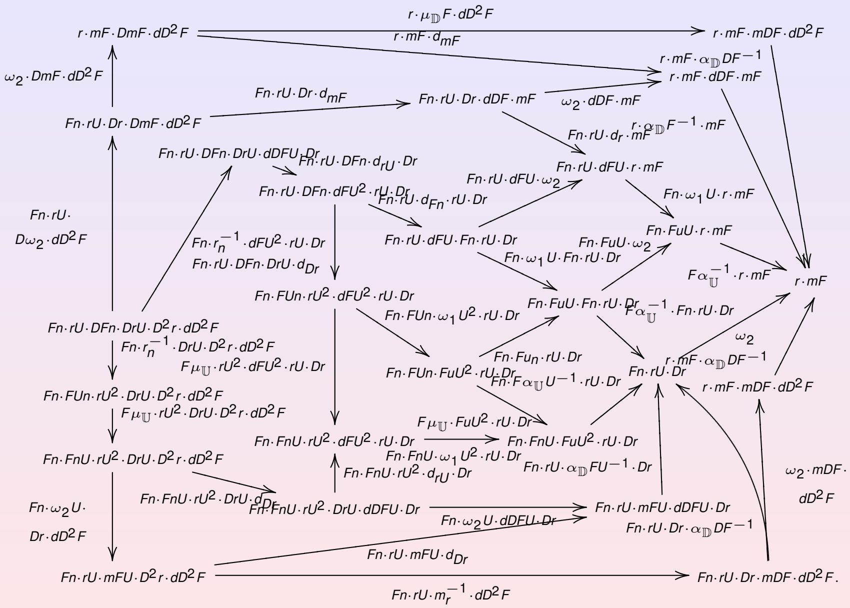
 & \searrow^{\,Fu\mathbf{A}} & FU\mathbf{A}
\end{array}
$$

Transitions for right Kan pseudomonads

$\omega_2\mathbf{A}$ is the unique 2-cell such that



Proof of the second coherence condition:



Pseudo-Distributive laws

A distributive law of $\mathbb{U}$ over $\mathbb{D}$ consists of

a transition $(r : UD \to DU, \omega_1, \omega_3)$ from $\mathbb{U}$ to $\mathbb{U}$ along D,

together with an op-transition (r, ω_2, ω_4) from $\mathbb{D}$ to $\mathbb{D}$ along U

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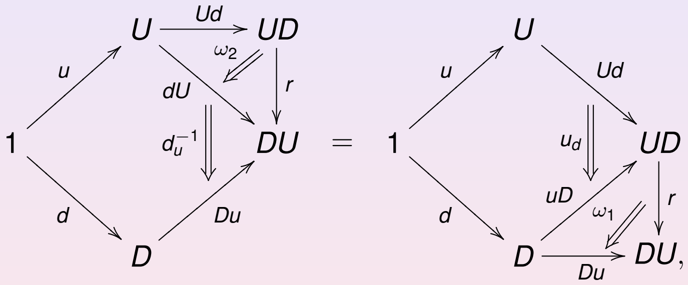

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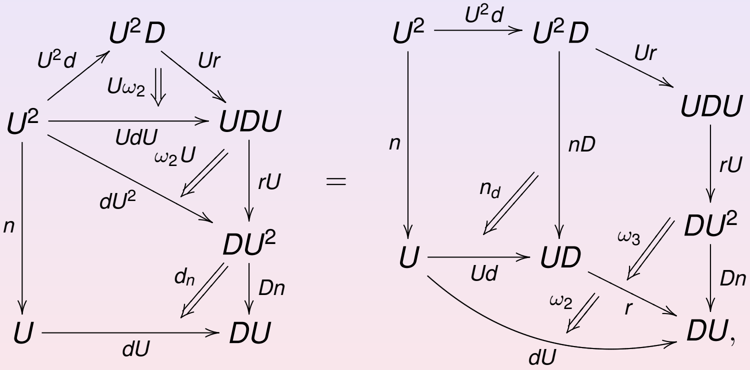

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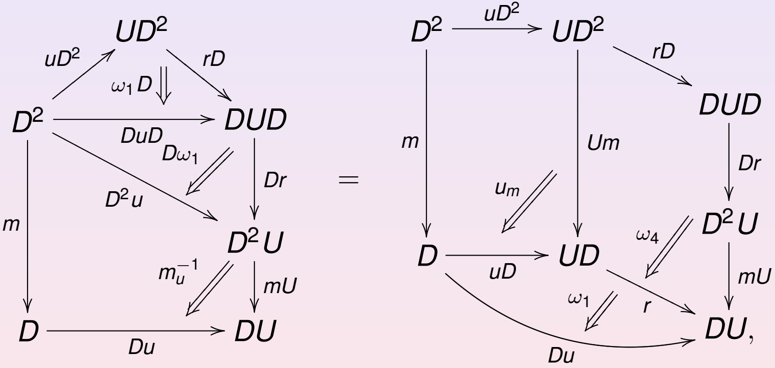

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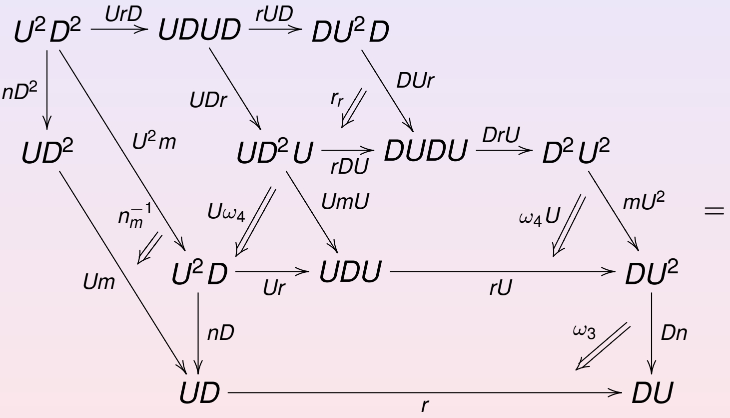
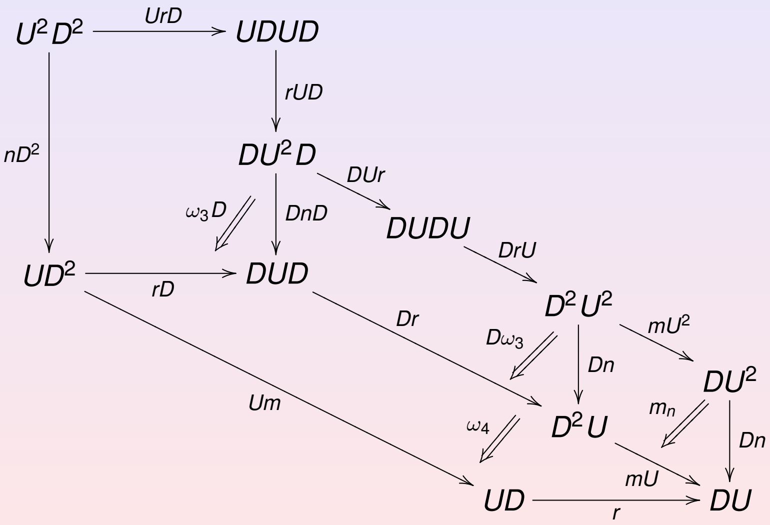


Distributive laws



Pseudo Distributive Laws

Lemma

$\mathbb{D}$ be a pseudomonad on $\mathcal{K}$

$\mathbb{U}$ be a colax idempotent pseudomonad on $\mathcal{K}$.

If there is a distributive law of $\mathbb{U}$ over $\mathbb{D}$, then

- *For every $\mathbf{A}$,*

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{u\mathbf{A}} & U\mathbf{A} \\ {\scriptstyle d\mathbf{A}}\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle d_{u\mathbf{A}}^{-1}} & \downarrow{\scriptstyle dU\mathbf{A}} \\ D\mathbf{A} & \xrightarrow[Du\mathbf{A}]{} & DU\mathbf{A} \end{array}$$

  exhibits $dU\mathbf{A}$ as a right Kan extension of $Du\mathbf{A} \cdot d\mathbf{A}$ along $u\mathbf{A}$.

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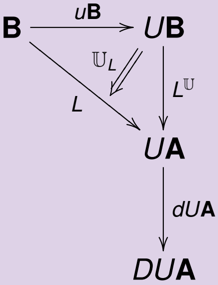
Pseudo Distributive Laws

Lemma

- *For every $L : \mathbf{B} \to U\mathbf{A}$ in $\mathcal{K}$,*



exhibits $dU\mathbf{A} \cdot L^{\mathbb{U}}$ as a right Kan extension of $dU\mathbf{A} \cdot L$ along $u\mathbf{B}$.

Pseudo Distributive Laws

Theorem

$\mathbb{U}$ is a colax idempotent pseudomonad on $\mathcal{K}$

$\mathbb{D}$ is a lax idempotent pseudomonad on $\mathcal{K}$

such that the conditions of the previous Lemma are satisfied.

Then a distributive law of $\mathbb{U}$ over $\mathbb{D}$ can be given by the following data:

- *For every $\mathbf{A}$ in $\mathcal{K}$, a $\mathbb{U}$-algebra structure $(D U\mathbf{A}, (\ )^{\lambda})$,*

such that the following two conditions are satisfied:

- *For every $L : \mathbf{B} \to U\mathbf{A}$, $D(L^{\mathbb{U}}) : (DU\mathbf{B}, (\ )^{\lambda}) \to (DU\mathbf{A}, (\ )^{\lambda})$ is 1-cell of $\mathbb{U}$-algebras.*
- *For every $H : \mathbf{C} \to DU\mathbf{A}$, $(H^{\lambda})^{\mathbb{D}} : (DU\mathbf{C}, (\ )^{\lambda}) \to (DU\mathbf{A}, (\ )^{\lambda})$ is an algebra morphism.*

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