On (binary) localic products and localic groups

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PORTUGAL

— joint work with Aleš Pultr (Charles University, Prague, CZ)

THE SETTING locales (or frames)

- Complete lattices L satisfying

$$\boxed{a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)}$$

(= complete Heyting algebras)

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$$\begin{tikzcd} L \arrow[d, "f"'] \\ M \end{tikzcd} \qquad \begin{tikzcd} L \\ M \arrow[u, "f^*"'] \end{tikzcd}$$

preserves $\bigvee$ (incl. the bottom 0)

$\wedge$ (incl. the top 1)

Loc Frm

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$$\begin{tikzcd} L \cong \mathbf{2} \times L \arrow[r, "\varepsilon\times\mathrm{id}"] \arrow[dr, equal] & L \times L \arrow[d, "\mu"] & L \times \mathbf{2} \cong L \arrow[l, "\mathrm{id}\times\varepsilon"'] \arrow[dl, equal] \\ & L & \end{tikzcd}$$

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GROUPS IN Loc: LOCALIC GROUPS

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$\boxed{aa^{-1} = \varepsilon = a^{-1}a}$

THE CATEGORY OF LOCALIC GROUPS

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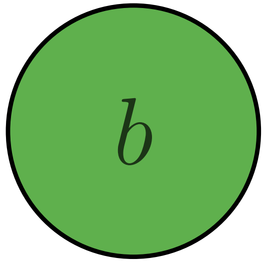

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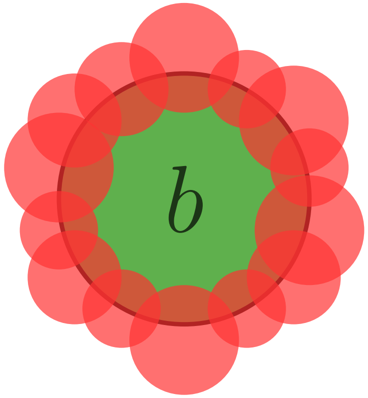

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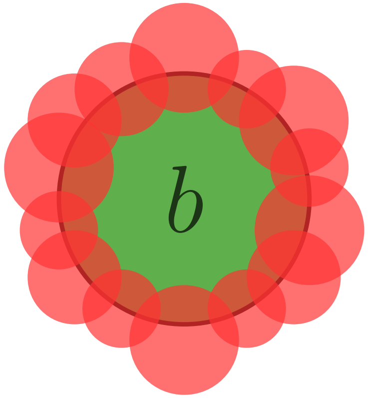

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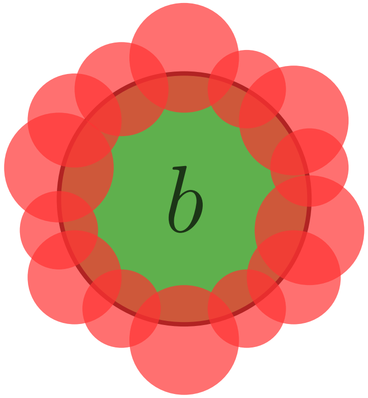

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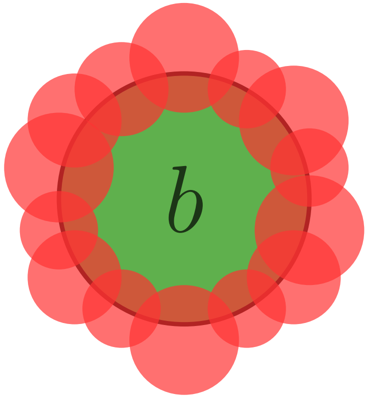

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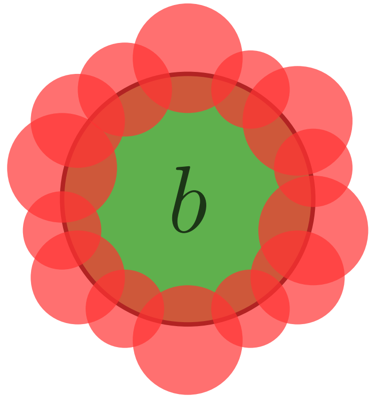

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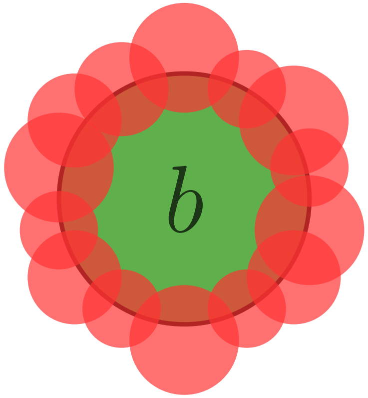

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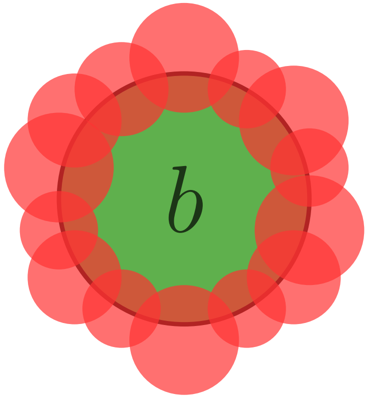

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- (U1) for every $U \in \mathscr{U}$ there is a $V \in \mathscr{U}$ such that $VV \leqslant U$,
- (U2) for each $a \in L$, $a = \bigvee \{b \in L \mid b \lhd_{\mathscr{U}} a\}$.

Category of uniform locales UCLoc

Uniform maps: $f : (L, \mathscr{U}) \to (M, \mathscr{V})$

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$f^*[V] \in \mathscr{U}$ for all $V \in \mathscr{V}$

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(LEFT and RIGHT) UNIFORMITIES ON LOCALIC GROUPS

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On the completeness of localic groups, **CMUC 40 (1999) 293-307**

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QUESTION: are $L \longmapsto (L, \mathscr{U}_l(L))$ and $L \longmapsto (L, \mathscr{U}_r(L))$ functorial ?

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$$\boxed{\nabla_L(E) = 1}$$

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$$\bigvee\{a \mid a \otimes a \leqslant E\}$$

$$\shortparallel$$

$$\boxed{\nabla_L(E) = 1}$$

UNIFORMITIES (Weil type)

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- (E3) for each $a \in L$, $a = \bigvee\{b \in L \mid b \lhd_{\mathscr{E}} a\}$.

Category of uniform locales

$\mathsf{U_E Loc}$

Uniform maps: $f : (L, \mathscr{E}) \to (M, \mathscr{F})$

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THEOREM.

The categories U$_E$Loc and U$_C$Loc are concretely isomorphic.

(Surprising, since Ω does not preserve products.)

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Nice features of localic products

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Nice features of localic products

1. $a \otimes b \leqslant E_U, b \neq 0 \implies a \leqslant Ub$.

2. $0 \neq a \otimes b \underbrace{\leqslant E}_{\text{symmetric}} \implies (a \vee b) \otimes (a \vee b) \leqslant E \circ E.$

Under this isomorphism:

UNIFORMITIES ON LOCALIC GROUPS

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Loc

$$\begin{tikzcd} L \times L \arrow[d, "\mathrm{id}\times\iota"'] & L \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "!"] & L \times L \arrow[d, "\iota\times\mathrm{id}"] \\ & \mathbf{2} \arrow[d, "\varepsilon"] & \\ L \times L \arrow[r, "\mu"'] & L & L \times L \arrow[l, "\mu"] \end{tikzcd}$$

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L \otimes L \arrow[r, "\nabla"] & L & L \otimes L \arrow[l, "\nabla"'] \\
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 & E_l(a) \qquad 1 & & & \\
 & & \mathbf{2} \arrow[uu, "!"'] & & \\
 & a \in \mathcal{N} \arrow[u, squiggly] & & \varepsilon^*(a) = 1 & \\
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JP & A. PULTR
Entourages, covers and localic groups, **Appl. Categ. Struct., to appear**

QUESTION ON FUNCTORIALITY

PROPOSITION. Each LG-map $f : (L, \mu_L, \varepsilon_L, \iota_L) \to (M, \mu_M, \varepsilon_M, \iota_M)$ is uniform w.r.t. both the left and right uniformities.

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$$a \rightsquigarrow E_l(a) \in \mathscr{E}_l(M)$$

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& a \arrow[rr, squiggly] \arrow[d, squiggly] & & E_l(a) \in \mathscr{E}_l(M) & \\
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$$\begin{tikzcd}
M \arrow[rr, "\mu_M^*"] \arrow[ddd, "f^*"'] & & M\otimes M \arrow[rr, "1_M\otimes\iota_M^*"] \arrow[ddd, "f^*\otimes f^*"] & & M\otimes M \arrow[ddd, "f^*\otimes f^*"] \\
& a \arrow[rr, squiggly] \arrow[d, squiggly] & & E_l(a)\in\mathscr{E}_l(M) & \\
& f^*(a) \arrow[rr, squiggly] & & E_l(f^*(a))\in\mathscr{E}_l(L) & \\
L \arrow[rr, "\mu_L^*"'] & & L\otimes L & & L\otimes L \arrow[ll, "1_L\otimes\iota_L^*"]
\end{tikzcd}$$