Quotients in hyperdoctrines

G. Rosolini
joint work with Maria Emilia Maietti

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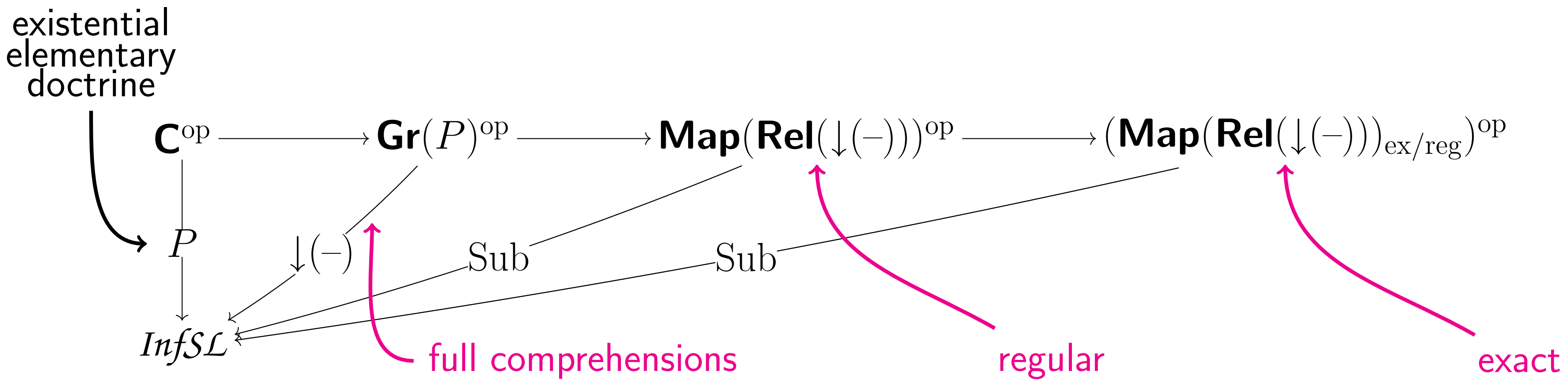
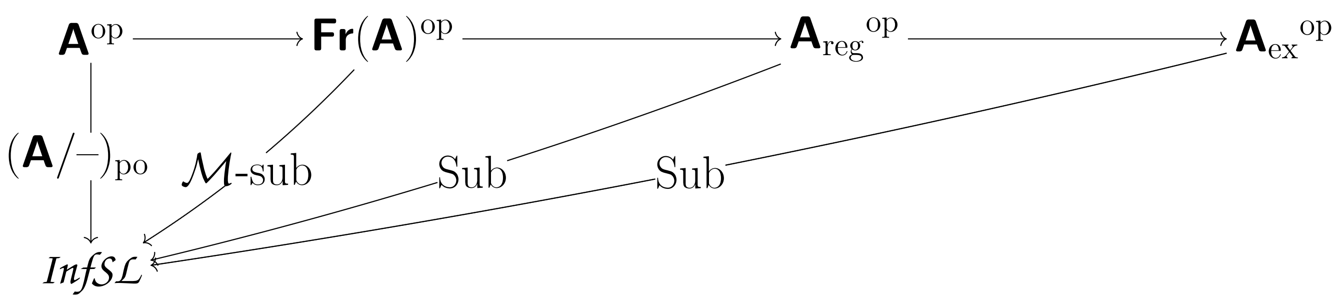
Completing a preordered fibration to a subobject fibration



For **A** with finite products and weak equalizers



Equivalence relations in elementary doctrines

An **elementary** doctrine is $\mathbf{C}^{\text{op}} \xrightarrow{P} \mathcal{I}nf\mathcal{SL}$ such that

- $\mathbf{C}$ has binary products
- for each map $\mathrm{id}_X \times \Delta_A \colon X \times A \to X \times A \times A$ in $\mathbf{C}$,
  the functor $P_{\mathrm{id}_X \times \Delta_A} \colon P(X \times A \times A) \to P(X \times A)$ has a left adjoint $\exists_{\mathrm{id}_X \times \Delta_A}$ which satisfy

  Beck-Chevalley Condition: for any arrow $f \colon A' \to A$—producing a pullback diagram

$$\begin{array}{ccc} X' \times A & \xrightarrow{\mathrm{id}_{X'} \times \Delta_A} & X' \times A \times A \\ {\scriptstyle f \times \mathrm{id}_A} \downarrow & & \downarrow {\scriptstyle f \times \mathrm{id}_A \times \mathrm{id}_A} \\ X \times A & \xrightarrow{\mathrm{id}_X \times \Delta_A} & X \times A \times A \end{array}$$

  —for any β, the natural map

$$\exists_{\mathrm{id}_{X'} \times \Delta_A} P_{f \times \mathrm{id}_A \times \mathrm{id}_A}(\beta) \leq P_{f \times \mathrm{id}_A} \exists_{\mathrm{id}_X \times \Delta_A}(\beta) \quad \text{is iso}$$

  Frobenius Reciprocity: for α in $P(X \times A \times A)$, β in $P(X \times A)$, the natural map

$$\exists_{\mathrm{id}_X \times \Delta_A}(P_{\mathrm{id}_X \times \Delta_A}(\alpha) \wedge \beta) \leq \alpha \wedge \exists_{\mathrm{id}_X \times \Delta_A}(\beta) \quad \text{is iso.}$$

Equivalence relations in elementary doctrines

An **elementary** doctrine is $\mathbf{C}^{\mathrm{op}} \xrightarrow{P} \mathcal{InfSL}$ such that

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A P-**equivalence relation over** A is an object ρ in $P(A \times A)$ such that

$$\delta_A \le \rho$$

$$\boxed{\delta_A := \exists_{\Delta_A}(\top_A)}$$

$$\rho \le P_{\langle \mathrm{pr}_2, \mathrm{pr}_1 \rangle}(\rho)$$

$$P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\rho) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\rho) \le P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\rho)$$

Examples

- $\mathbf{X}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathit{Inf}\mathcal{SL}$: the subobject functor on a category $\mathbf{X}$ with binary products and pullbacks

- $\mathbf{C}^{\mathrm{op}} \xrightarrow{T} \mathit{Inf}\mathcal{SL}$: a tripos on a category $\mathbf{C}$ with finite products

- $\mathbf{V}^{\mathrm{op}} \xrightarrow{LT} \mathit{Inf}\mathcal{SL}$: the Lindenbaum-Tarski algebras of well-formed formulae of a first order theory on the category $\mathbf{V}$ of lists of typed variables and substitutions

- $\mathbf{A}^{\mathrm{op}} \xrightarrow{(\mathbf{A}/-)_{\mathrm{po}}} \mathit{Inf}\mathcal{SL}$: the poset reflections of the comma categories on a category $\mathbf{A}$ with binary products and weak pullbacks

- $\mathbf{ML}^{\mathrm{op}} \xrightarrow{P^{ML}} \mathit{Inf}\mathcal{SL}$: the functor of propositions in context of Martin-Löf type theory on the category $\mathbf{ML}$ of closed types and terms in context

Quotients

A **quotient** of a P-equivalence relation ρ in $P(A \times A)$ is an arrow $q\colon A \to C$ in **C** such that

- $\rho \leq P_{q\times q}(\delta_C)$ and
  for every $h\colon A \to X$ in **C** such that $\rho \leq P_{h\times h}(\delta_X)$
  there is a unique $k\colon C \to X$ in **C** such that $k \circ q = h$
- $P_{q\times q}(\delta_C) \leq \rho$
- there is a bijection between $P(C)$ and $\mathcal{D}es_\rho$, the sub-order of $P(A)$ on those α such that $P_{\mathrm{pr}_1}(\alpha) \wedge \rho \leq P_{\mathrm{pr}_2}(\alpha)$

It is **stable** if, for $g\colon B \to C$, then there is a pullback

$$\begin{array}{ccc} P & \xrightarrow{q'} & B \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[q]{} & C \end{array}$$

and $q'\colon P \to B$ is a quotient of $P_{g'\times g'}(\rho)$

Trying to add quotients to an elementary doctrine

$$\mathbf{Q}_P{}^{\mathrm{op}} \xrightarrow{\ \overline{P}\ } \mathcal{InfSL}$$

an object of $\mathbf{Q}_P$ is (A, ρ) such that A is an object in $\mathbf{C}$ and ρ in $P(A \times A)$ is an equivalence relation

an arrow of $\mathbf{Q}_P$ is $\big[f\big]\colon (A,\rho) \to (B,\sigma)$ is an equivalence class of arrows $f\colon A \to B$ in $\mathbf{C}$
such that $\quad \rho \le P_{f\times f}(\sigma)$
where $\quad f \sim g \quad$ when $\quad \rho \le_{A\times A} P_{f\times g}(\sigma)$

the semilattice $\overline{P}(A,\rho)$ is $\mathcal{D}es_\rho$.

The functor $\quad \begin{array}{ccc} \mathbf{C} & \xrightarrow{\ J\ } & \mathbf{Q}_P \\ A & \longmapsto & (A, \delta_A) \end{array} \quad$ is full and $\overline{P} \circ J = P$.

When $\mathbf{C}^{\mathrm{op}} \xrightarrow{\ P\ } \mathcal{InfSL}$ is $\mathbf{A}^{\mathrm{op}} \xrightarrow{\ (\mathbf{A}/-)_{\mathrm{po}}\ } \mathcal{InfSL}$

$\mathbf{Q}_P{}^{\mathrm{op}} \xrightarrow{\ \overline{P}\ } \mathcal{InfSL}$ is $\mathbf{A}_{\mathrm{ex}}{}^{\mathrm{op}} \xrightarrow{\ \mathrm{Sub}\ } \mathcal{InfSL}$

Comprehensions

For α in $P(A)$, a **weak comprehension of** α is an arrow $\{|\alpha|\}\colon X \to A$ in $\mathbf{C}$ such that

- $\top_X \leq P_{\{|\alpha|\}}(\alpha)$
- for every arrow $g\colon Y \to A$ such that $\top_Y \leq P_g(\alpha)$ there is a (not necessarily unique) $h\colon Y \to X$ such that

$$g = \{|\alpha|\} \circ h.$$

It is **stable** when, for every arrow $f\colon A' \to A$ in $\mathbf{C}$, $P_{f'}(\alpha)$ has a weak comprehension and there is a weak pullback

$$\begin{array}{ccc} X' & \xrightarrow{\{|P_f(\alpha)|\}} & A' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[\{|\alpha|\}]{} & A. \end{array}$$

Say that $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathcal{InfSL}$ is **extensional** if Δ_A is a comprehension of δ_A for every A in $\mathbf{A}$.

Results

For an elementary doctrine $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$

- If $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ has weak comprehensions
  then $\overline{P}\colon {\mathbf{Q}_P}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ is an elementary doctrine with stable quotients and (strict) comprehensions.
  Moreover $\mathbf{Q}_P$ is a regular category.

- If $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ is extensional then $J\colon P \to \overline{P}$ is faithful.
  Moreover comprehensions are full in P if and only if they are so in $\overline{P}$.

- The assignment $P \mapsto \overline{P}$ determines a left biadjoint to the inclusion of
  extensional elementary doctrines with quotients and comprehensions
  into extensional elementary doctrines with comprehensions.

- If $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ is existential with weak comprehensions
  then $\overline{P}\colon {\mathbf{Q}_P}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ is existential.

- If $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ has full weak comprehensions and
  - for every arrow f in $\mathbf{C}$, the functor P_f has a right adjoint and these satisfy Beck-Chevalley condition
  - $\mathbf{C}$ is weakly cartesian closed

  then $\mathbf{Q}_P$ is cartesian closed.

Comparing the two completions

For an elementary doctrine $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathcal{InfSL}$

$$
\begin{array}{ccccc}
\mathbf{C}^{\mathrm{op}} & \longrightarrow & \mathbf{Q}_P & \longrightarrow & (\mathbf{Map}(\mathbf{Rel}(\downarrow(-)))_{\mathrm{ex/reg}})^{\mathrm{op}} \\
 & {\scriptstyle P}\searrow & \downarrow{\scriptstyle \overline{P}} & \swarrow{\scriptstyle \mathrm{Sub}} & \\
 & & \mathcal{InfSL} & &
\end{array}
$$