A Topological Theory of ($\mathbb{T}$,V)-Categories

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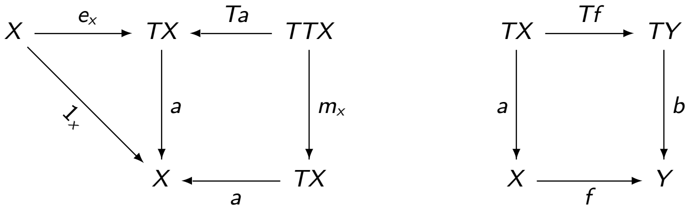
1. $(\mathbb{T},V)$-Categories
2. L-closure
3. L-compactness
4. L-separation
5. L-completeness

- $\mathbb{T} = (T, e, m)$ on Set

  Eilenberg-Moore algebra $(X, a : TX \to X)$



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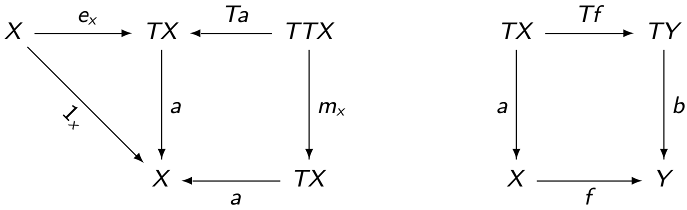

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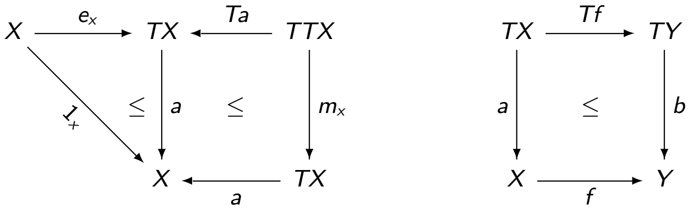

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Strict Topological Theory (Hofmann 2007)

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- $\mathbb{T} = (T, e, m)$: T, m satisfy BC, $(T1 = 1)$
- V quantale
- $\xi : TV \to V$ compatible with $\mathbb{T}$ and V
  - $1_V = \xi . e_V$
  - $\xi . T\xi = \xi . m_V$
  - $k.! = \xi . Tk$
  - $\otimes . < \xi . T\pi_1, \xi . T\pi_2 >= \xi . T(\otimes)$
  - $(\xi_x)_X : P_V \to P_V T$ nat. trans.

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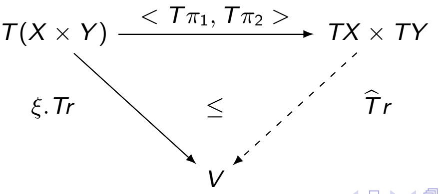
- $r : X \nrightarrow Y$



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$\mathbb{T} = \mathbb{I} \implies V$-enriched categories

$V = 2 \implies$ <u>Ord</u>

$V = \mathbb{P}_+ = [0, \infty]^{op} \implies$ <u>Met</u>

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$\mathbb{T} = U,\ V = \mathbb{P}_+ \implies$ App (Clementino & Hofmann, 2003)

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  Closed maps: $\mathcal{F} = \{f : X \to Y \mid f(\overline{M}^{\mathscr{L}}) = \overline{f(M)}^{\mathscr{L}}\}$

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$$\textbf{Met}, \qquad y \in \overline{M} \iff 0 \geq d(y, M) = \inf_{z \in M} d(y, z)$$

$$\textbf{V-Cat}, \qquad y \in \overline{M} \iff k \leq \bigvee_{z \in M} a(y, z)$$

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$y \in \overline{M}^{\mathscr{L}} \iff k \leq \bigvee_{\mathfrak{x} \in TM} a(\mathfrak{x}, y) \otimes$ **?**

L-closure

- $A \dashv S : (\mathbb{T}, V)\text{-Cat} \to V\text{-Cat}$

  ($Specialization$:$\underline{\text{Top}} \to \underline{\text{Ord}}$, $Alexandroff$:$\underline{\text{Ord}} \to \underline{\text{Top}}$)

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$\underline{\text{Top}}$ (b-closure)

$y \in \overline{M}^{b} \iff \forall U$ open nbhd of y, $U \cap M \cap \overline{\{y\}} \neq \emptyset$

$\underline{\text{App}}$ (Zariski closure, Giuli 2006)

$y \in \overline{M}^{Z} \iff \forall \alpha, \beta \in \mathscr{R}\ (\alpha_{|M} = \beta_{|M} \Rightarrow \alpha(y) = \beta(y))$

(X, d), $y \in \overline{M}^{Z} \iff \forall \varepsilon > 0,\ d(y, M \cap \{y\}^{(\varepsilon)}) = 0$

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X is L-compact $\iff \mathscr{L}(X)$ is compact

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Examples

<u>Top</u>	b-topology of X is compact
<u>App</u>	Zariski compact ?

L-compactness

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$$y \in \overline{M} := \quad k \leq \bigvee_{\mathfrak{x} \in TM} a(\mathfrak{x}, y) \qquad \Longrightarrow \tau$$

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$$\begin{array}{ccc} \underline{App} & \xrightarrow{\;B\;} & \underline{BiTop} \\ & \searrow^{\mathscr{L}} & \downarrow J \\ & & \underline{Top} \end{array}$$

$$B(X) := (X, \tau, \tau^{d}), \quad J(X, \tau, \tau^{d}) := (X, \tau \vee \tau^{d})$$

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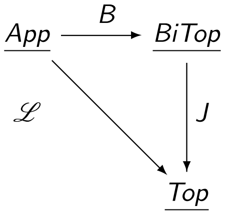

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App

X Zariski compact $\iff$ i) Every τ-closed set is τ^d-compact
ii) Every τ^d-closed set is τ-compact

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- $$\varphi : (X,a) \rightharpoonup (Y,b) \ (\mathbb{T},\mathsf{V})\text{-module} :\iff \begin{array}{l} \varphi : TX \nrightarrow Y \\ \varphi \circ a = \varphi \ \& \ b \circ \varphi = \varphi \end{array}$$

$$f : (X,a) \to (Y,b) \implies \begin{array}{ll} f_* : X \rightharpoonup Y, & f_*(\mathfrak{x},y) = b(Tf(\mathfrak{x}),y) \\ f^* : Y \rightharpoonup X, & f^*(\eta,x) = b(\eta,f(x)) \end{array}$$

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Examples

<u>Top</u> X is T_0

<u>App</u> Top. coreflection of X is T_0

L-completeness

Lawvere (1973)

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Top

X L-complete $\Longleftrightarrow$ X weakly sober

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Definition (L-complete $(\mathbb{T},\mathrm{V})$ functor)

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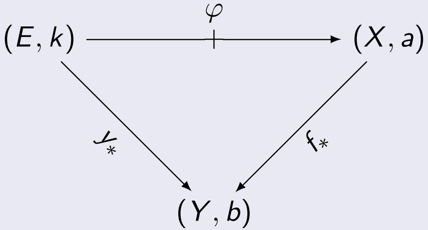

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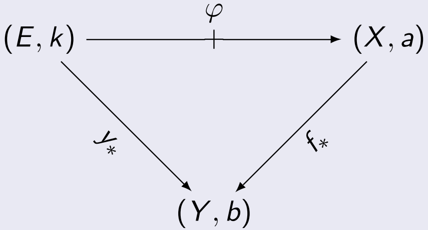

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- (X,a) L-complete $\Longleftrightarrow$ $!_X : (X,a) \longrightarrow (1,\top)$ L-complete

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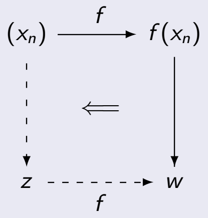

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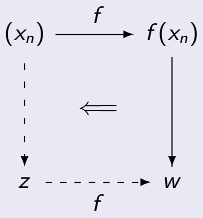

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Properties

- Pullback stable
- X L-complete, Y L-sep. $\Longrightarrow \forall f : X \to Y$ L-complete
- Cancellation w.r.t. L-separated maps, $f : X \to Y$ L-sep. $\Longleftrightarrow \forall x, z \in X \; (x_* = z_* \; \& \; f(x) = f(z) \Rightarrow x = z)$

Factorization System

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- β : *Tych* $\rightarrow$ *CpctHaus*

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- $\mathcal{Y} : (\mathbb{T}, \mathsf{V})\text{-}\mathbf{Cat} \to (\mathbb{T}, \mathsf{V})\text{-}\mathbf{Cat}_{\text{cpl \& sep}}$

  Factorization Sys. on $(\mathbb{T},\mathsf{V})$-**Cat** : $(\mathcal{Y}^{-1}\{\mathrm{Iso}\}$, L-comp & L-sep)

  $\mathcal{Y}^{-1}\{\mathrm{Iso}\} = \{f \mid f_* \circ f^* = 1,\ f^* \circ f_* = 1\}$