Alexander Campbell
Macquarie University

Enriched algebraic weak factorisation systems

A modification of Garner's small object argument shows that if $\mathcal{V}$ is a monoidal model category in which every object is cofibrant, then any cofibrantly generated $\mathcal{V}$-enriched model category has a cofibrant replacement $\mathcal{V}$-comonad and a fibrant replacement $\mathcal{V}$-monad [4]. Conversely, an elementary argument shows that if a monoidal model category $\mathcal{V}$ with cofibrant unit object has a cofibrant replacement $\mathcal{V}$-comonad, then every object of $\mathcal{V}$ is cofibrant [2].

These results leave open the following question: what extra structure, if not an enrichment in the ordinary sense, is naturally possessed by the (co)fibrant replacement (co)monad of an enriched model category when not every object of the base monoidal model category is cofibrant? The purpose of this talk is to answer this question.

An analysis of the monoidal model category **2-Cat** of 2-categories (in which not every object is cofibrant, and which is monoidal under the Gray tensor product) suggests the decisive concept. For while the cofibrant replacement comonad **st** on **2-Cat**, which sends a 2-category A to its pseudofunctor classifier $\mathbf{st}A$, fails to extend to a **Gray**-comonad, it is nevertheless a monoidal closed comonad, and so comes equipped with pseudofunctors $\mathbf{Gray}(A, B) \longrightarrow \mathbf{Gray}(\mathbf{st}A, \mathbf{st}B)$ enriching **st** with the structure of a "locally weak **Gray**-comonad". Generally, given a monoidal/closed comonad Q on a monoidal/closed category $\mathcal{V}$, one can define a 2-category of $\mathcal{V}$-categories, "locally Q-weak $\mathcal{V}$-functors", and "locally Q-weak $\mathcal{V}$-natural transformations".

Abstracting from these observations, I will introduce notions of monoidal, closed, and enriched algebraic weak factorisation systems (which are strengthenings of the notions of bi(co)lax morphisms of AWFS [3]) and demonstrate that the cofibrant replacement comonad Q for a monoidal/closed AWFS (L, R) on a monoidal/closed category $\mathcal{V}$ is a monoidal/closed comonad on $\mathcal{V}$, and that the (co)fibrant replacement (co)monad for an (L, R)-enriched AWFS (H, M) on a $\mathcal{V}$-category $\mathcal{A}$ is a locally Q-weak $\mathcal{V}$-(co)monad on $\mathcal{A}$, and moreover that the category of weak maps [1] for (H, M) is enriched over the skew-monoidal/closed category of weak maps for (L, R).

References:

[1] John Bourke and Richard Garner. Algebraic weak factorisation systems II: Categories of weak maps. *J. Pure Appl. Algebra* 220 (2016), no. 1, 148–174.

[2] Stephen Lack and Jiří Rosický. Homotopy locally presentable enriched categories. *Theory Appl. Categ.* 31 (2016), no. 25, 712–754.

[3] Emily Riehl. Monoidal algebraic model structures. *J. Pure Appl. Algebra* 217 (2013), no. 6, 1069–1104.

[4] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.