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Towards a categorification of integers

The motivation comes from Stephen Schanuel's question:

"Where are negative sets?

Though ill-posed, the question is suggestive; a good answer should complete the diagram

$$\begin{array}{ccc} \mathbb{S} & \hookrightarrow & \mathbb{E} \\ \downarrow & & \downarrow \\ \mathbb{N} & \hookrightarrow & \mathbb{Z} \end{array}$$

where $\mathbb{S}$ is the category of finite sets; we seek an enlargement $\mathbb{E}$, the isomorphism classes of which should give rise to all integers, rather than just natural numbers ([4])."

We would like to present a background for constructing a positive answer to the above question, based on generalized multisets. A multiset is a set with multiple elements. The first known observation that one can define a generalized multiset with arbitrary integer (positive, negative or zero) multiplicities, belongs to Hassler Whitney ([5]). Systematic studies in this field started with the works of Wolfgang Reisig ([3], ch. 9), Wayne D. Blizard ([1]) and Daniel Loeb ([2]).

When we restrict multiplicities to: 1, 0, -1, we obtain a generalized set which is a pair of disjoint sets (A, B), where A is the positive part and B is the negative one. Generalized union and intersection are defined by max and min of multiplicities, respectively, so

$$(A, B) \overset{g}{\cup} (C, D) = (A \cup C, B \cap D), \;\; (A, B) \overset{g}{\cap} (C, D) = (A \cap C, B \cup D).$$

Inclusion is defined by inequality between multiplicities, so

$$(A, B) \overset{g}{\subset} (C, D) \Leftrightarrow A \subset C, \; D \subset B.$$

If A and B are finite disjoint sets, we put $|A| - |B|$ to be the generalized cardinality of (A, B). Natural candidates for a direct sum and a direct product of (A, B) and (C, D) are:

$$(A \sqcup C, B \sqcup D), \;\; (A \times C \sqcup B \times D, A \times D \sqcup B \times C).$$

Now, we can precise Schanuel's question if it is possible to define in some natural way maps between finite generalized sets in order to obtain a category extending the category of finite sets. It may be also interesting to look for some similar constructions in other categories, where two pairs of objects (A, B) and (C, D) are isomorphic if and only if $A \oplus D$ and $B \oplus C$ are isomorphic in the initial category.

References:

[1] W.D. Blizard, Negative membership, *Notre Dame J. Formal Logic* 31 (1990) 346–368.

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[3] W. Reisig, Petri nets, An introduction, *Monogr. Theoret. Comput. Sci. EATCS* 4, Springer, Berlin, 1985.

[4] S. Schanuel, Negative sets have Euler characteristic and dimension, in: Category Theory, Como 1990, *Lecture Notes in Math.* 1488, Springer, Berlin, 1991, 379–385.

[5] H. Whitney, Characteristic functions and the algebra of logic, *Ann. of Math.* 34 (1933) 405–414.