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The Wasserstein monad in categorical probability

In existing approaches to categorical probability theory, one works with a suitable category of measurable spaces and equips it with a monad, which associates to every space X the space of probability measures on X. This applies e.g. to the Giry monad [1] or the Radon monad [2]; see also [3] for a more general setup. These monads constitute an extra piece of structure that needs to be put in by hand. Here, we introduce another such monad—the *Wasserstein monad*—and prove that it arises from a colimit construction on the underlying category CMS (compact metric spaces).

Besides the utility of this colimit characterization, an advantage of the Wasserstein monad over the existing ones is as follows. Deriving quantitative bounds on approximations is a standard tool in probability theory. Therefore we also expect that working with metric spaces will allow us more easily to find categorical proofs and perhaps generalizations of probability theory's basic results, such as the law of large numbers, or similarly the Glivenko-Cantelli theorem on the convergence of the empirical distribution.

Another advantage is that the Wasserstein monad is a monoidal monad with respect to the closed monoidal structure on CMS given by adding the distances [4, Section 2]; as one would expect, the monoidal structure encodes the formation of product distributions. The Giry monad on the category of measurable spaces does not have both properties: the category of measurable spaces is not cartesian closed; and while there is another monoidal structure with respect to which the category is closed, in this one the Giry monad does not even permit a strength [5], and therefore it lacks an essential piece of structure needed for probability theory.

References:

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