On *On the operads of J. P. May* of G. M. Kelly

Simon Cho

University of Pennsylvania

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Motivating example

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There are morally two ideas to codify:

- That the m different n_i-ary objects combine to give a single $n_1+\cdots+n_m$-ary object, and
- That this composition respects the internal symmetry of each n_i-ary object (more on this symmetry in the next slide)

Functors on the permutation category

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Thus $[A^{n_1}, A] \otimes \cdots \otimes [A^{n_m}, A]$ inherits the internal symmetry of each of its n_i-ary components. This is concisely formalized by the following:

Day convolution

Let $T, S \in \mathcal{F}$. We write:

$$T \otimes S = \int^{m,n} \mathbf{P}(m + n, -) \otimes Tm \otimes Sn$$

Day convolution

- Associativity:

$$T \otimes (S \otimes R) \simeq (T \otimes S) \otimes R \simeq \int^{m,n,k} \mathbf{P}(m+n+k, -) \otimes Tm \otimes Sn \otimes Rk$$

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- Symmetry:

$$\begin{array}{ccc} T_1 \otimes \cdots \otimes T_m & = & \int^{n_1,\ldots,n_m} \mathbf{P}(n_1 + \cdots + n_m, -) \otimes T_1 n_1 \otimes \cdots \otimes T_m n_m \\ \langle\xi\rangle \downarrow & & \downarrow \mathbf{P}(\langle\xi\rangle, -) \otimes \langle\xi\rangle \\ T_{\xi 1} \otimes \cdots \otimes T_{\xi m} & = & \int^{n_1,\ldots,n_m} \mathbf{P}(n_{\xi 1} + \cdots + n_{\xi m}, -) \otimes T_{\xi 1} n_{\xi 1} \otimes \cdots \otimes T_{\xi m} n_{\xi m} \end{array}$$

  - where $\langle\xi\rangle$ denotes either a contravariant action $n_{\xi 1} + \cdots + n_{\xi m} \to n_1 + \cdots + n_m$ on $\mathbf{P}$ or a covariant action $T_1 n_1 \otimes \cdots \otimes T_m n_m \to T_{\xi 1} n_{\xi 1} \otimes \cdots \otimes T_{\xi m} n_{\xi m}$ on $\mathcal{V}$.

Day convolution, cont'd

Some technical details to keep in mind:

- The last point of the previous slide shows us that $(m, T) \mapsto T^m$ gives us a functor $\mathbf{P}^{\mathrm{op}} \times \mathcal{F} \to \mathcal{F}$

Day convolution, cont'd

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- The last point of the previous slide shows us that $(m, T) \mapsto T^m$ gives us a functor $\mathbf{P}^{\mathrm{op}} \times \mathcal{F} \to \mathcal{F}$
- $\mathcal{F}$ is a $\mathcal{V}$-category, with $\mathcal{V}$-hom objects given by $[T, S] = \int_n [Tn, Sn]$ for $T, S \in \mathcal{F}$
- $\mathcal{V}$ includes into $\mathcal{F}$ as a full coreflective subcategory, preserving monoidal structure
    - We can therefore write $\mathcal{F}(A \otimes T, S) \simeq \mathcal{V}(A, [T, S])$

Combining arities

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Substitution product

Let $T, S \in \mathcal{F}$. We define:

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- If $S \in \mathcal{V}$ then $T \circ S \in \mathcal{V}$
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- $J = \mathbf{P}(1, -) \otimes I$ is the identity for $\circ$
- For $S \in \mathcal{F}$, $- \circ S : \mathcal{F} \to \mathcal{F}$ has the right adjoint $\{S, -\}$ given by

$$\{S, R\}m = [S^m, R]$$

- If $A \in \mathcal{V}$ then $\mathcal{V}(T \circ A, B) \simeq \mathcal{F}(T, \{A, B\})$

This notation for the right adjoint agrees with our previous choice to denote $n \mapsto [A^n, A]$ by $\{A, A\}$.

The endomorphism operad

Definition: operads

An *operad* is a monoid for $\circ$.

- That is, some $T \in \mathcal{F}$ with data $\mu : T \circ T \to T$ and $\eta : J \to T$ satisfying the monoid axioms.
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Recall that $\{A, A\} \in \mathcal{F}$ is given by $\{A, A\}n = [A^n, A]$.

- The composite $\{A, A\} \circ \{A, A\} \circ A \xrightarrow{1 \circ e} \{A, A\} \circ A \xrightarrow{e} A$ gives us, by adjunction:
$$\mu : \{A, A\} \circ \{A, A\} \to \{A, A\}$$
- $J \circ A \simeq A$ corresponds by adjunction to $\eta : J \to \{A, A\}$

Thus $\{A, A\}$ is an operad (the *endomorphism operad*)!

A closer look

We explicitly describe

$$[A^m, A] \otimes ([A^{n_1}, A] \otimes \cdots \otimes [A^{n_m}, A]) \to [A^{n_1+\cdots+n_m}, A]$$

in terms of the operad structure on $\{A, A\}$ (for tidiness we assume $m = 2$):

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in terms of the operad structure on $\{A, A\}$ (for tidiness we assume $m = 2$):

$$\begin{array}{ccccc} ([A^{n_1}, A] \otimes A^{n_1}) & & ([A^{n_2}, A] \otimes A^{n_2}) & & \\ \downarrow e & \otimes & \downarrow e & \otimes [A^2, A] \xrightarrow{e} A \\ A & & A & & \end{array}$$

gives us

$$[A^2, A] \otimes ([A^{n_1}, A] \otimes [A^{n_2}, A]) \otimes A^{n_1+n_2} \to A$$

which corresponds by adjunction to

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This is exactly the construction of the operad map for $\{A, A\}$.

Operad algebras

$\{A, A\}$ is the "universal" operad, in the following sense:

- Every operad $T \in \mathcal{F}$ gives a monad $T \circ -$ on $\mathcal{F}$ (or on $\mathcal{V}$ by restriction).
- Given $A \in \mathcal{F}$, algebra structures $h' : T \circ A \to A$ for the monad $T \circ -$ on A correspond precisely to operad morphisms $h : T \to \{A, A\}$

In the above case we say that h gives an algebra structure on A for the operad T.

Thank you!

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