Generalizing Lawvere theories

(an exposition on *Notions of Lawvere Theories* by Lack & Rosicky)

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what do we hope to achieve?

Goals of Notions – generalize it

Three tracks of generalization

– replace finite products with another class

– replace the base category in which models are taken

– everything enriched

Properties of classical Lawvere theory to retain

– Lawvere theory - monad correspondence

– algebraic functors have left adjoints

– reflectiveness of models

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the terminology of presentability

The terminology of presentability

Let Φ be a class of limits

(def) A $\mathbf{C}$-object x is Φ-**presentable** if

$$\mathbf{C}(x, -)\colon \mathbf{C} \to \mathbf{Set}$$

preserves any colimit that commutes with limits in Φ.

Denote the full subcategory of Φ-presentable objects by $\mathbf{C}_\Phi$.

replacing finite products

replacing finite products – setup

We are now entering the world of $\mathcal{V}$-categories where

- $\mathcal{V}$ is symmetric closed monoidal plus complete and co-complete
- Φ is a class of **weights**,
  ↳ functors $J^{\mathrm{op}} \to \mathcal{V}$ required to define enriched (co)limits

(?) In order to construct an analogy

classical Lawvere theory	$\leftrightarrow$	**general Lawvere theory**
finite products		Φ-limits

what should Φ-limits be?

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(axiom A) All Φ-limits commute in $\mathcal{V}$ with colimits that have Φ-continuous weights.

This is an abstraction of

- finite limits commuting with filtered colimits in **Set**

What Φ satisfies axiom A?

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Given a collection $\mathbb{D}$ of small categories that satisfy some axioms, we can construct viable Φ.

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Instead, here is an example

(ex) ($\mathbb{D}$:= finite categories) $\Rightarrow$ (Φ = enriched finite limits)

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(ex) ($\mathbb{D} :=$ finite categories) $\quad\Rightarrow\quad$ ($\Phi =$ enriched finite limits)

a correspondence between Lawvere Φ-theories & monads

Lawvere Φ-theories & monads – setting the stage

So far, we have

replaced ordinary categories with $\mathcal{V}$-categories

— and —

replaced finite products with Φ-limits via axiom A

To obtain the **Lawvere theory-monad correspondence**, we generalize the base category by assuming...

(axiom B) fix a $\mathcal{V}$-category $\mathcal{K}$ with finite limits and is equivalent to Φ-**Cts**$(\mathcal{K}_\Phi, \mathcal{V})$

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Lawvere Φ-theories & monads – three categories

(def) A Lawvere Φ-theory is a $\mathcal{V}$-functor $\mathcal{K}_\Phi^{\mathrm{op}} \to \mathcal{T}$ that is bijective-on-objects and Φ-continuous

$$\mathbf{Law}_\Phi(\mathcal{K}) := \begin{cases} \textbf{(objects)} & \text{Lawvere } \Phi\text{-theories} \\ \textbf{(morphisms)} & \text{commuting triangles} \end{cases}$$

For a Lawvere theory $\mathcal{K}_\Phi^{\mathrm{op}} \to \mathcal{T}$

$$\mathbf{Mod}_\Phi(\mathcal{T}, \mathcal{K}) := \begin{cases} \textbf{(objects)} & \mathcal{V}\text{-functors } \mathcal{T} \to \mathcal{K} \text{ s.t.} \\ & \mathcal{K}_\Phi^{\mathrm{op}} \to \mathcal{T} \to \mathcal{K} \text{ is } \Phi\text{-continuous.} \\ \textbf{(morphisms)} & \text{appropriate natural transformations} \end{cases}$$

$$\mathbf{Mnd}_\Phi(\mathcal{K}) := \begin{cases} \textbf{(objects)} & \mathcal{V}\text{-monads on } \mathcal{K} \text{ that preserve} \\ & \Phi\text{-flat colimits.} \\ \textbf{(morphisms)} & \text{appropriate natural transformations} \end{cases}$$

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Lawvere Φ-theories & monads – the equivalence

First, we define a functor $\mathrm{mnd}\colon \mathbf{Law}_\Phi(\mathcal{K}) \to \mathbf{Mnd}_\Phi(\mathcal{K})$

(Prop 6.6 in Lack & Rosicky)

$$\begin{array}{ccc} \mathbf{Mod}_\Phi(\mathcal{T},\mathcal{K}) & \longrightarrow & [\mathcal{T},\mathcal{V}] \\ \downarrow u & & \downarrow [j,V] \\ \mathcal{K} \xrightarrow{\;y\;} [\mathcal{K}^{\mathrm{op}},\mathcal{V}] & \xrightarrow{\;\iota^*\;} & [\mathcal{K}_\Phi^{\mathrm{op}},\mathcal{V}] \end{array}$$

The diagram is a pullback and u is monadic via a monad m that preserves Φ-flat colimits.

Define $\mathrm{mnd}(\mathcal{T}) := m$.

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Let's define another functor $\mathrm{th}\colon \mathbf{Mnd}_\Phi(\mathcal{K}) \to \mathbf{Law}_\Phi(\mathcal{K})$

(1st) Factor $m \in \mathbf{Mnd}_\Phi(\mathcal{K})$ through the EM-category

$$\mathcal{K} \to \mathcal{K}^m \to \mathcal{K}$$

(2nd) Restrict the first $\mathcal{V}$-functor to $\mathcal{K}_\Phi$

$$f\colon \mathcal{K}_\Phi \to \mathcal{K} \to \mathcal{K}^m$$

(3rd) Note that f factors (uniquely up to unique iso)

Define $\mathrm{th}(m) := (\mathcal{K}_\Phi^{\mathrm{op}} \xrightarrow{\ell^{\mathrm{op}}} \mathcal{G}^{\mathrm{op}})$.

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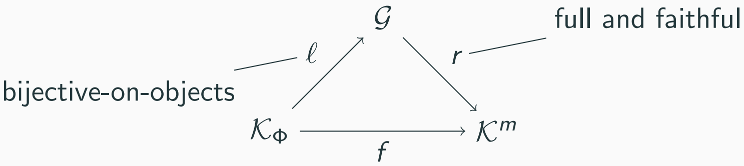

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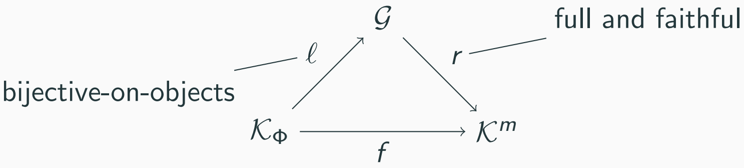

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A theorem

There is an equivalence

$$\mathbf{Law}_\Phi(\mathcal{K}) \underset{\mathrm{th}}{\overset{\mathrm{mnd}}{\rightleftarrows}} \mathbf{Mnd}_\Phi(\mathcal{K}) \qquad \simeq$$

adjoints to algebraic functors & reflectivity

- We continue with $\mathcal{V}$-categories and a class of limits Φ satisfying axiom A
- We replace our base category given by axiom B with...

> **(axiom C)** fix a $\mathcal{V}$-category $\mathcal{K}$ with Φ-limits such that $y \colon \mathcal{K} \to [\mathcal{K}^{\mathsf{op}}, \mathcal{V}]$ has a Φ-continuous left adjoint.

Axiom C holds for presheaf categories and, when $\mathcal{V} = \mathbf{Set}$, Grothendieck topoi.

adjoints and reflectivity – Φ-theories

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adjoints and reflectivity – a new notion of theory

(def) A Φ-theory is a small $\mathcal{V}$-category with Φ-limits

In our new context, we have the following two categories

$$\Phi\text{-}\mathbf{Th} := \begin{cases} \textbf{(objects)} & \Phi\text{-theories} \\ \textbf{(morphisms)} & \text{evident functors} \end{cases}$$

$$\Phi\text{-}\mathbf{Mod}(\mathcal{T}) := \Phi\text{-}\mathbf{Cts}(\mathcal{T}, \mathcal{K})$$

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adjoints and reflexivity – algebraic functors

(def) An algebraic functor is the pullback functor

$$g^*\colon \Phi\text{-}\mathbf{Cts}(\mathcal{T},\mathcal{K}) \to \Phi\text{-}\mathbf{Cts}(\mathcal{S},\mathcal{K})$$

induced from a morphism of Φ-theories $g\colon \mathcal{S} \to \mathcal{T}$.

These have left adjoints constructed from left Kan extensions.

Theorem 5.1 in Lack & Rosicky

The map $F \mapsto \operatorname{Lan}_g F$ gives a functor

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Here's a depiction of the scenario:

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$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\quad F \quad} & \mathcal{K} \\ & {\scriptstyle g}\searrow \quad \nearrow {\scriptstyle g_*(F) = \mathrm{Lan}_g F} & \\ & \mathcal{T} & \end{array}$$

A theorem

A Φ-theory morphism $g\colon \mathcal{S} \to \mathcal{T}$ induces an adjunction

$$\Phi\text{-}\mathbf{Cts}(\mathcal{T}, \mathcal{K}) \underset{g_*}{\overset{g^*}{\rightleftarrows}} \Phi\text{-}\mathbf{Cts}(\mathcal{S}, \mathcal{K}) \quad (\bot)$$

adjoints and reflectivity – models are reflective

Theorem 5.2 in Lack and Rosicky

Φ-$\mathbf{Cts}(\mathcal{T},\mathcal{K})$ is reflective in $[\mathcal{T},\mathcal{K}]$

(pf)

- Φ-$\mathbf{Cts}(\mathcal{F}\mathcal{T},\mathcal{K}) \simeq [\mathcal{T},\mathcal{K}]$ from the free/forgetful adjunction

$$\mathcal{V}\text{-}\mathbf{Cat} \underset{\mathcal{U}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{V}\text{-}\mathbf{Cat}(\Phi\text{-lim}) \quad (\mathcal{F} \dashv \mathcal{U})$$

- $\mathcal{T}$ has Φ-limits, hence $\mathcal{T} \hookrightarrow \mathcal{F}\mathcal{T}$ has a right adjoint r.

- The induced algebraic functor

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