Dense generators of algebraic categories

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Dense generators

Definition

We say a full small subcategory $\mathcal{A} \hookrightarrow \mathcal{C}$ is a dense generator of $\mathcal{C}$ if its associated nerve functor $\nu_{\mathcal{A}} : \mathcal{C} \to PSh(\mathcal{A})$ is fully faithful, where $\nu_{\mathcal{A}}(X) = \mathcal{C}(\imath_{\mathcal{A}}(-), X)$ for each $X \in \mathcal{C}$.

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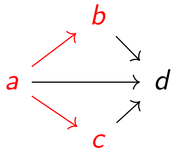
Example

- 

;
- $\aleph_0 \hookrightarrow \mathbf{Set}$;
- $\mathbf{KHaus}$ has no dense generator (Gabriel, Ulmer 1971);
- The full subcategory containing only $\mathbb{R} \oplus \mathbb{R}$ is dense in $\mathbf{Vect}_{\mathbb{R}}$.

Dense generators

The nerve functor associated to $\mathcal{A} \hookrightarrow \mathcal{C}$ has a left adjoint $PSh(\mathcal{A}) \to \mathcal{C}$, which takes F to the weighted colimit of $i_{\mathcal{A}}$ with weight F. If $\mathcal{A}$ is a dense generator, this canonically makes every object X of $\mathcal{C}$ a colimit of objects in $\mathcal{A}$.

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Compare this to the fact that presheaves $F : \mathcal{D}^{op} \to \mathbf{Set}$ over small categories are canonically colimits of representables, as in the coend identity

$$F(-) = \int^{d \in \mathcal{D}} F(d) \times \mathcal{D}(-, d).$$

Theories with arity

Motivation: if $I : \aleph_0 \to \mathcal{L}$ is a Lawvere theory, the restriction functor $I^* : PSh(L) \to PSh(\aleph_0)$ induces a monad $I^* I_!$ that preserves the essential image of the nerve functor.

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Definition

Let $\mathcal{C}$ be a category with a dense generator $\mathcal{A}$. A theory with arities $\mathcal{A}$ on $\mathcal{C}$ is a is a pair (Θ, j), where $j : \mathcal{A} \to \Theta$ is a bijective-on-objects functor such that the induced monad $j^* j_!$ on $PSh(\mathcal{A})$ preserves the essential image of $\nu_{\mathcal{A}}$. A Θ-model is a presheaf on Θ whose restriction along j is isomorphic to some $\mathcal{A}$-nerve.

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For $\mathcal{A} = \aleph_0 \hookrightarrow \mathbf{Set}$, this requirement on models says a Θ-model M restricts to powers of some object: $Mj(-) \cong X^{|-|}$ for some set X.

Monads with arities

Let T be a monad on a category $\mathcal{C}$ with a dense generator $\mathcal{A}$. Then it is a monad with arities $\mathcal{A}$ if the composite $\nu_{\mathcal{A}} T$ maps the $\mathcal{A}$-cocones of $\mathcal{C}$ into colimits in $PSh(\mathcal{A})$.

Main results

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- The Nerve Theorem: if $\mathcal{C}$ has a dense generator $\mathcal{A}$ and T is a monad with arities $\mathcal{A}$, then $\mathcal{C}^T$ has a dense generator consisting of the free T-algebras on objects of $\mathcal{A}$;
- Gabriel-Ulmer/Adámek-Rosický Theorem: if T is a monad on an α-accessible category that preserves α-filtered colimits, then its category of algebras is α-accessible and has a dense generator given by the free T-algebras on α-presentable objects.

Consequences

- Classical Lawvere theories are equivalent to finitary monads on $\mathbf{Set}$;

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- The free algebraic objects on finitely many generators give a dense generator of the corresponding algebraic category;

Thanks for your time!

Example

- Both dagger categories and groupoids are models for algebraic theories over the category $i\mathbf{Grph}$ of involutive graphs;

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- The free dagger categories/groupoids on (finite, connected) acyclic graphs form a dense generator: a cocontinuous functor is thus determined by the images of the free objects on acyclic graphs.

Example

- The subcategory of graphs of the form $0 \rightleftarrows 1 \rightleftarrows \ldots \rightleftarrows n$ is another dense generator of $i\mathbf{Grph}$;

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- The free dagger categories on such sequences form a dense generator of $\mathbf{DagCat}$. Any cocontinuous functor on $\mathbf{DagCat}$ is thus determined by the images of dagger categories of the form

$$0 \underset{f_1}{\overset{(f_1)^\dagger}{\rightleftarrows}} 1 \underset{f_2}{\overset{(f_2)^\dagger}{\rightleftarrows}} \dots \underset{f_n}{\overset{(f_n)^\dagger}{\rightleftarrows}} n \; .$$