Towards a Characterization of the Double Category of Spans

Evangelia Aleiferi

Dalhousie University

Category Theory 2017
July, 2017

Motivation

Theorem (Lack, Walters, Wood 2010)

For a bicategory $\mathcal{B}$ the following are equivalent:

i. *There is an equivalence $\mathcal{B} \simeq \mathcal{S}pan(\mathcal{E})$, for some finitely complete category $\mathcal{E}$.*

ii. *$\mathcal{B}$ is Cartesian, each comonad in $\mathcal{B}$ has an Eilenberg-Moore object and every map in $\mathcal{B}$ is comonadic.*

iii. *The bicategory $\mathcal{M}ap(\mathcal{B})$ is an essentially locally discrete bicategory with finite limits, satisfying in $\mathcal{B}$ the Beck condition for pullbacks of maps, and the canonical functor $C : \mathcal{S}pan(\mathcal{M}ap(\mathcal{B})) \to \mathcal{B}$ is an equivalence of bicategories.*

Cartesian Bicategories

Definition (Carboni, Kelly, Walters, Wood 2008)

A bicategory $\mathcal{B}$ is said to be **Cartesian** if:

i. The bicategory $\mathcal{M}ap(\mathcal{B})$ has finite products
ii. Each category $\mathcal{B}(A, B)$ has finite products.
iii. Certain derived lax functors $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $I : \mathbb{1} \to \mathcal{B}$, extending the product structure of $\mathcal{M}ap(\mathcal{B})$, are pseudo.

Cartesian Bicategories

Definition (Carboni, Kelly, Walters, Wood 2008)

A bicategory $\mathcal{B}$ is said to be **Cartesian** if:

i. The bicategory $\mathcal{M}ap(\mathcal{B})$ has finite products

ii. Each category $\mathcal{B}(A, B)$ has finite products.

iii. Certain derived lax functors $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $I : \mathbb{1} \to \mathcal{B}$, extending the product structure of $\mathcal{M}ap(\mathcal{B})$, are pseudo.

Examples

1. The bicategory $\mathcal{R}el(\mathcal{E})$ of relations over a regular category $\mathcal{E}$.

Cartesian Bicategories

Definition (Carboni, Kelly, Walters, Wood 2008)

A bicategory $\mathcal{B}$ is said to be **Cartesian** if:

i. The bicategory $\mathcal{M}ap(\mathcal{B})$ has finite products
ii. Each category $\mathcal{B}(A, B)$ has finite products.
iii. Certain derived lax functors $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $I : \mathbb{1} \to \mathcal{B}$, extending the product structure of $\mathcal{M}ap(\mathcal{B})$, are pseudo.

Examples

1. The bicategory $\mathcal{R}el(\mathcal{E})$ of relations over a regular category $\mathcal{E}$.
2. The bicategory $\mathcal{S}pan(\mathcal{E})$ of spans over a finitely complete category $\mathcal{E}$.

Question

For a finitely complete category $\mathcal{E}$, can we characterize the double category $\mathbb{S}pan(\mathcal{E})$ of

- objects of $\mathcal{E}$
- arrows of $\mathcal{E}$ vertically
- spans in $\mathcal{E}$ horizontally

as a Cartesian double category?

Cartesian double categories

Definition

A **(pseudo) double category** $\mathbb{D}$ consists of:

i. A category D_0, representing the objects and the vertical arrows.

ii. A category D_1, representing the horizontal arrows and the cells written as

$$\begin{array}{ccc} A & \overset{M}{\nrightarrow} & B \\ f\downarrow & \alpha & \downarrow g \\ C & \underset{N}{\nrightarrow} & D \end{array}$$

iii. Functors $D_1 \xrightarrow{S,T} D_0$, $D_0 \xrightarrow{U} D_1$ and $D_1 \times_{D_0} D_1 \xrightarrow{\odot} D_1$ such that $S(U_A) = A = T(U_A)$, $S(M \odot N) = SN$ and $T(M \odot N) = TM$.

iv. Natural isomorphisms $(M \odot N) \odot P \xrightarrow{a} M \odot (N \odot P)$,

$U_{TM} \odot M \xrightarrow{l} M$ and $M \odot U_{SM} \xrightarrow{r} M$, satisfying the pentagon and triangle identities.

The objects, the horizontal arrows and the cells with source and target identities form a bicategory

$$\mathcal{H}(\mathbb{D}).$$

The objects, the horizontal arrows and the cells with source and target identities form a bicategory

$$\mathcal{H}(\mathbb{D}).$$

The double categories together with the double functors and the vertical natural transformations form a 2-category

$$\mathbf{DblCat}.$$

Definition

A double category $\mathbb{D}$ is said to be **Cartesian** if there are adjunctions

$$\mathbb{D} \underset{\times}{\overset{\Delta}{\bot}} \mathbb{D}\times\mathbb{D} \quad \text{and} \quad \mathbb{D} \underset{I}{\overset{!}{\bot}} \mathbb{1} \quad \text{in } \mathbf{DblCat}.$$

Definition

A double category $\mathbb{D}$ is said to be **Cartesian** if there are adjunctions

$$\mathbb{D} \underset{\times}{\overset{\Delta}{\rightleftarrows}} \mathbb{D} \times \mathbb{D} \quad (\Delta \dashv \times) \quad \text{and} \quad \mathbb{D} \underset{I}{\overset{!}{\rightleftarrows}} \mathbb{1} \quad (! \dashv I) \quad \text{in } \mathbf{DblCat}.$$

Examples

1. The double category $\mathbb{R}el(\mathcal{E})$ of relations over a regular category $\mathcal{E}$

Definition

A double category $\mathbb{D}$ is said to be **Cartesian** if there are adjunctions

$$\mathbb{D} \underset{\times}{\overset{\Delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{D}\times\mathbb{D} \quad \text{and} \quad \mathbb{D} \underset{I}{\overset{!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{1} \quad \text{in } \mathbf{DblCat}.$$

Examples

1. The double category $\mathbb{R}el(\mathcal{E})$ of relations over a regular category $\mathcal{E}$
2. The double category $\mathbb{S}pan(\mathcal{E})$ of spans over a finitely complete category $\mathcal{E}$

Definition

A double category $\mathbb{D}$ is said to be **Cartesian** if there are adjunctions

$$\mathbb{D} \underset{\times}{\overset{\Delta}{\rightleftarrows}} \mathbb{D}\times\mathbb{D} \quad (\Delta \dashv \times) \qquad \text{and} \qquad \mathbb{D} \underset{I}{\overset{!}{\rightleftarrows}} \mathbb{1} \quad (! \dashv I)$$

in **DblCat**.

Examples

1. The double category $\mathbb{R}el(\mathcal{E})$ of relations over a regular category $\mathcal{E}$
2. The double category $\mathbb{S}pan(\mathcal{E})$ of spans over a finitely complete category $\mathcal{E}$
3. The double category $V - \mathbb{M}at$, for a Cartesian monoidal category V with coproducts such that the tensor distributes over them

Definition

A double category is called **fibrant** if for every niche of the form

$$\begin{array}{ccc} A & & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \underset{M}{\longrightarrow\!\!\!\!\!\!\!+\;\,} & D \end{array}$$

there is a horizontal arrow $g^*Mf_* : A \nrightarrow C$ and a cell

$$\begin{array}{ccc} A & \xrightarrow{g^*Mf_*} & C \\ {\scriptstyle f}\downarrow & \zeta & \downarrow{\scriptstyle g} \\ B & \underset{M}{\longrightarrow} & D, \end{array}$$

so that every cell $\begin{array}{ccc} A' & \xrightarrow{M'} & C' \\ {\scriptstyle fh}\downarrow & & \downarrow{\scriptstyle gk} \\ B & \underset{M}{\longrightarrow} & D, \end{array}$ can be factored uniquely through ζ.

Examples

Examples

1. $\mathbb{R}el(\mathcal{E})$, $\mathcal{E}$ a regular category

Examples

1. $\mathbb{R}el(\mathcal{E})$, $\mathcal{E}$ a regular category
2. $\mathbb{S}pan(\mathcal{E})$, $\mathcal{E}$ finitely complete category

Examples

1. $\mathbb{R}el(\mathcal{E})$, $\mathcal{E}$ a regular category
2. $\mathbb{S}pan(\mathcal{E})$, $\mathcal{E}$ finitely complete category
3. $V - \mathbb{M}at$, V a Cartesian monoidal category with coproducts such that the tensor distributes over them

Theorem

Consider a fibrant double category $\mathbb{D}$ such that:

i. *the vertical category D_0 has finite products $\times, p, r, I$ and*
ii. *every $\mathcal{H}(\mathbb{D})(A, B)$ has finite products $\wedge, \top$.*

Then the formula $M \times N = (p^* M p_*) \wedge (r^* N r_*)$ and the terminal horizontal arrow $\top_{I,I}$ extend the product of D_0 to lax double functors

$$\times : \mathbb{D} \times \mathbb{D} \to \mathbb{D} \text{ and } I : \mathbb{1} \to \mathbb{D}.$$

Theorem

Consider a fibrant double category $\mathbb{D}$ such that:

i. *the vertical category D_0 has finite products $\times, p, r, I$ and*
ii. *every $\mathcal{H}(\mathbb{D})(A, B)$ has finite products $\wedge, \top$.*

Then the formula $M \times N = (p^* M p_*) \wedge (r^* N r_*)$ and the terminal horizontal arrow $\top_{I,I}$ extend the product of D_0 to lax double functors

$$\times : \mathbb{D} \times \mathbb{D} \to \mathbb{D} \text{ and } I : \mathbb{1} \to \mathbb{D}.$$

Theorem

Consider a fibrant double category $\mathbb{D}$ such that:

i. *the vertical category D_0 has finite products $\times, p, r, I$,*
ii. *every $\mathcal{H}(\mathbb{D})(A, B)$ has finite products $\wedge, \top$ and*
iii. *the lax double functors $\times$ and I are pseudo.*

Then $\mathbb{D}$ is Cartesian.

Eilenberg-Moore Objects

Consider the double category $\mathbb{C}om(\mathbb{D})$ with:

- Objects the comonads in $\mathbb{D}$: $(X, P: X \nrightarrow X)$, equipped with globular cells $\delta : P \to P \odot P$ and $\epsilon : P \to U_X$ satisfying the usual conditions.

Consider the double category $\mathbb{C}om(\mathbb{D})$ with:

- Objects the comonads in $\mathbb{D}$: $(X, P : X \nrightarrow X)$, equipped with globular cells $\delta : P \to P \odot P$ and $\epsilon : P \to U_X$ satisfying the usual conditions.
- Vertical arrows the comonad morphisms, i.e. vertical arrows $f : X \to Y$ together with a cell ψ:

$$\begin{array}{ccc} X & \overset{P}{\nrightarrow} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \underset{R}{\nrightarrow} & Y \end{array}$$

  which is compatible with the comonad structure.
- Horizontal arrows the horizontal comonad maps, i.e. horizontal arrows $F : X \nrightarrow X'$, together with a cell α:

$$\begin{array}{ccccc} X & \overset{P}{\nrightarrow} & X & \overset{F}{\nrightarrow} & X' \\ \| & & & & \| \\ X & \underset{F}{\nrightarrow} & X' & \underset{P'}{\nrightarrow} & X', \end{array}$$

  compatible with the counit and the comultiplication.

- Cells that are compatible with the horizontal and the vertical structure.

- Cells that are compatible with the horizontal and the vertical structure.

Definition

We say that the double category $\mathbb{D}$ **has Eilenberg-Moore objects for comonads** if the inclusion double functor

$$J : \mathbb{D} \to \mathbb{C}om(\mathbb{D})$$

has a right adjoint.

Example

The double category $\mathbb{S}pan(\mathcal{E})$, for a finitely complete category $\mathcal{E}$, has Eilenberg-Moore objects for comonads.

Definition

If a double category $\mathbb{D}$ has Eilenberg-Moore objects, then for every comonad (X, P) there is an object EM and a universal comonad morphism

$$\begin{array}{ccc} EM & \overset{U}{\longrightarrow} & EM \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e} \\ X & \underset{P}{\longrightarrow} & X. \end{array}$$

If $\mathbb{D}$ is fibrant and $P \cong e^* e_*$, we say that $\mathbb{D}$ has **strong Eilenberg-Moore objects**.

Towards the characterization of spans

Definition

We say that a fibrant and Cartesian double category has the **Frobenius property** if for every object X and its pullback diagram of vertical arrows

$$
\begin{array}{ccc}
X & \xrightarrow{d} & XX \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d1} \\
XX & \xrightarrow[1d]{} & XXX
\end{array}
$$

the cell

$$
\begin{array}{ccccccc}
XX & \overset{d^*}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & X & \overset{U}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & X & \overset{d_*}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & XX \\
\| & & \downarrow{\scriptstyle d} & & {\scriptstyle d}\downarrow & & \| \\
XX & \overset{U}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & XX & & XX & \overset{U}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & XX \\
\| & & {\scriptstyle 1d}\downarrow & & \downarrow{\scriptstyle d1} & & \| \\
XX & \underset{(1d)_*}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & XXX & \underset{U}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & XXX & \underset{(d1)^*}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & XX
\end{array}
$$

is invertible.

Definition (Pare, Grandis)

A fibrant double category $\mathbb{D}$ **has tabulators** if for every horizontal arrow $F : X \nrightarrow Y$ there is an object T and a cell

$$
\begin{array}{ccc}
T & \overset{U}{\longrightarrow} & T \\
{\scriptstyle t_1}\downarrow & \tau & \downarrow{\scriptstyle t_2} \\
X & \underset{F}{\longrightarrow} & Y
\end{array}
$$

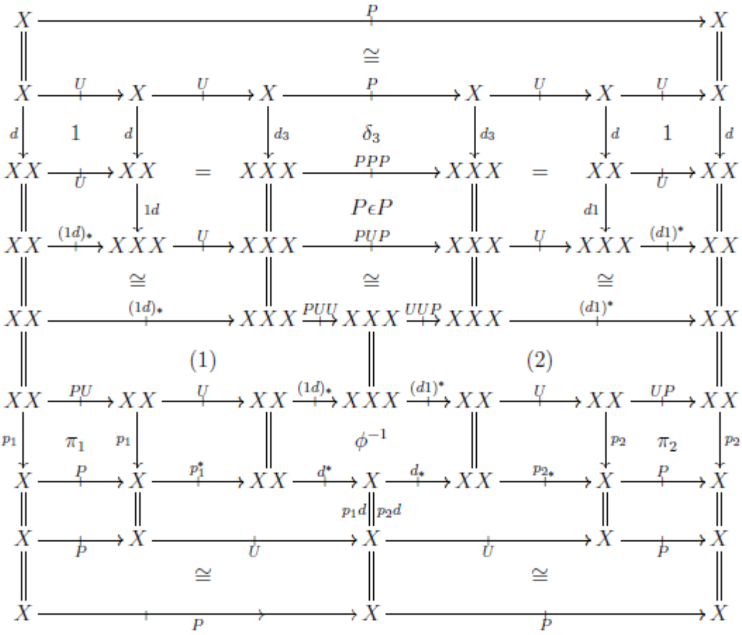
such that for every other object H and every cell $\beta : U_H \to F$, there is a unique vertical arrow $f : H \to T$ such that $\beta = \tau U_f$.
We say that the tabulators are **strong** if $F \cong t_{2*} t_1^*$.

Conjecture

If a double category is fibrant, Cartesian and it satisfies the Frobenius property, then for every cell of the form

$$\begin{array}{ccc} X & \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\;} & X \\ \Big\| & \epsilon & \Big\| \\ X & \underset{U_X}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\;} & X \end{array}$$

there exists a comonad structure (P, ϵ, δ)*, unique up to isomorphism.*


X P X
≅
X U X U X P X U X U X
d 1 d d3 δ3 d3 d 1 d
XX U XX = XXX PPP XXX = XX U XX
1d PϵP d1
XX (1d)* XXX U XXX PUP XXX U XXX (d1)* XX
≅ ≅ ≅
XX (1d)* XXX PUU XXX UUP XXX (d1)* XX
(1) (2)
XX PU XX U XX (1d)* XXX (d1)* XX U XX UP XX
p1 π1 p1 ϕ−1 p2 π2 p2
X P X p1* XX d* X d* XX p2* X P X
p1d p2d
X P X U X U X P X
≅ ≅
X P X P X

Corollary (of the conjecture)

In a fibrant and Cartesian double category $\mathbb{D}$, that satisfies the Frobenius property, for every horizontal arrow $F: X \nrightarrow Y$, the cartesian filling of the niche

$$\begin{array}{ccc} X \times Y & & X \times Y \\ {\scriptstyle d\times Y}\downarrow & & \downarrow{\scriptstyle X\times d} \\ X \times X \times Y & \xrightarrow[X\times F\times Y]{} & X \times Y \times Y \end{array}$$

admits a comonad structure, unique up to isomorphism.

Corollary (of the conjecture)

A fibrant and Cartesian double category $\mathbb{D}$ that satisfies the Frobenius property and has Eilenberg-Moore objects, has tabulators.

Corollary (of the conjecture)

A fibrant and Cartesian double category $\mathbb{D}$ *that satisfies the Frobenius property and has Eilenberg-Moore objects, has tabulators.*
Moreover, if the Eilenberg-Moore objects are strong, the tabulators are strong.

Corollary (of the conjecture)

If a double category $\mathbb{D}$ *satisfies all of the following, then* $\mathbb{D} \simeq \mathbb{S}pan(\mathbb{D})$*:*

Corollary (of the conjecture)

If a double category $\mathbb{D}$ satisfies all of the following, then $\mathbb{D} \simeq \mathbb{S}pan(\mathbb{D})$:

- *$\mathbb{D}$ is fibrant and Cartesian.*

Corollary (of the conjecture)

If a double category $\mathbb{D}$ *satisfies all of the following, then* $\mathbb{D} \simeq \mathbb{S}pan(\mathbb{D})$*:*

- $\mathbb{D}$ *is fibrant and Cartesian.*
- $\mathbb{D}$ *satisfies the Frobenius property*

Corollary (of the conjecture)

If a double category $\mathbb{D}$ satisfies all of the following, then $\mathbb{D} \simeq \mathbb{S}pan(\mathbb{D})$:

- *$\mathbb{D}$ is fibrant and Cartesian.*
- *$\mathbb{D}$ satisfies the Frobenius property*
- *$\mathbb{D}$ has strong Eilenberg-Moore objects*

Corollary (of the conjecture)

If a double category $\mathbb{D}$ satisfies all of the following, then $\mathbb{D} \simeq \mathbb{S}pan(\mathbb{D})$:

- *$\mathbb{D}$ is fibrant and Cartesian.*
- *$\mathbb{D}$ satisfies the Frobenius property*
- *$\mathbb{D}$ has strong Eilenberg-Moore objects*
- *D_0 has pullbacks*

Corollary (of the conjecture)

If a double category $\mathbb{D}$ satisfies all of the following, then $\mathbb{D} \simeq \mathbb{S}pan(\mathbb{D})$:

- *$\mathbb{D}$ is fibrant and Cartesian.*
- *$\mathbb{D}$ satisfies the Frobenius property*
- *$\mathbb{D}$ has strong Eilenberg-Moore objects*
- *D_0 has pullbacks*
- *For every composable horizontal arrows F and G, the tabulator of $G \odot F$ is given by the pullback of the tabulators of F and G.*

Corollary (of the conjecture)

If a double category $\mathbb{D}$ satisfies all of the following, then $\mathbb{D} \simeq \mathbb{S}pan(\mathbb{D})$:

- *$\mathbb{D}$ is fibrant and Cartesian.*
- *$\mathbb{D}$ satisfies the Frobenius property*
- *$\mathbb{D}$ has strong Eilenberg-Moore objects*
- *D_0 has pullbacks*
- *For every composable horizontal arrows F and G, the tabulator of $G \odot F$ is given by the pullback of the tabulators of F and G.*
- *For every pair of vertical arrows $r_0 : R \to A$ and $r_1 : R \to B$, the tabulator of $r_{1*} r_0^*$ is R.*

Further Questions

$\mathbf{FbrCat}^{\mathcal{Q}}$: The full sub-2-category of **DblCat** determined by the fibrant double categories $\mathbb{D}$ in which every category $\mathcal{H}(\mathbb{D})(A, B)$ has coequalizers and $\odot$ preserves them in both variables.

If $\mathbb{D}$ is a double category in $\mathbf{FbrCat}^{\mathcal{Q}}$, then we can define the double category $\mathbb{M}\mathbf{od}(\mathbb{D})$ of

- monads,
- monad morphisms vertically,
- modules horizontally and
- equivariant maps.

Moreover, $\mathbb{M}\mathbf{od}(\mathbb{D})$ is fibrant.

Theorem (Shulman 2007)

$\mathbb{M}$**od** *defines a 2-functor* $\textbf{FbrCat}^{\mathcal{Q}} \to \textbf{FbrCat}^{\mathcal{Q}}$.

Theorem (Shulman 2007)

$\mathbb{M}\mathbf{od}$ *defines a 2-functor* $\mathbf{FbrCat}^{\mathcal{Q}} \to \mathbf{FbrCat}^{\mathcal{Q}}$.

Since every 2-functor preserves adjunctions, we have the following:

Corollary

If $\mathbb{D}$ *is a fibrant Cartesian double category in which every category* $\mathcal{H}(\mathbb{D})(A, B)$ *has coequalizers and* $\odot$ *preserves them in both variables, then* $\mathbb{M}\mathbf{od}(\mathbb{D})$ *is a fibrant Cartesian double category too.*

Theorem (Shulman 2007)

$\mathbb{M}\mathbf{od}$ *defines a 2-functor* $\mathbf{FbrCat}^{\mathcal{Q}} \to \mathbf{FbrCat}^{\mathcal{Q}}$.

Since every 2-functor preserves adjunctions, we have the following:

Corollary

If $\mathbb{D}$ *is a fibrant Cartesian double category in which every category* $\mathcal{H}(\mathbb{D})(A, B)$ *has coequalizers and* $\odot$ *preserves them in both variables, then* $\mathbb{M}\mathbf{od}(\mathbb{D})$ *is a fibrant Cartesian double category too.*

In particular, for a finitely complete and cocomplete category $\mathcal{E}$, since

$$\mathbf{Prof}(\mathcal{E}) \simeq \mathbf{Mod}(\mathbf{Span}(\mathcal{E})),$$

the double category $\mathbf{Prof}(\mathcal{E})$ is Cartesian.

Theorem (Shulman 2007)

*$\mathbb{M}$**od** defines a 2-functor* $\mathbf{FbrCat}^{\mathcal{Q}} \to \mathbf{FbrCat}^{\mathcal{Q}}$.

Since every 2-functor preserves adjunctions, we have the following:

Corollary

If $\mathbb{D}$ is a fibrant Cartesian double category in which every category $\mathcal{H}(\mathbb{D})(A, B)$ has coequalizers and $\odot$ preserves them in both variables, then $\mathbb{M}\mathbf{od}(\mathbb{D})$ is a fibrant Cartesian double category too.

In particular, for a finitely complete and cocomplete category $\mathcal{E}$, since

$$\mathbf{Prof}(\mathcal{E}) \simeq \mathbf{Mod}(\mathbf{Span}(\mathcal{E})),$$

the double category $\mathbf{Prof}(\mathcal{E})$ is Cartesian.

Question

By using the above construction of modules, can we characterize the double category of profunctors as a Cartesian double category?

M Grandis and R Paré.
Limits in double categories.
Cahiers de topologie et géométrie différentielle catégoriques, 40(3):162–220, 1999.

Stephen Lack, R. F C Walters, and R. J. Wood.
Bicategories of spans as cartesian bicategories.
Theory and Applications of Categories, 24(1):1–24, 2010.

Susan Niefield.
Span, Cospan, and other double categories.
Theory and Applications of Categories, 26(26):729–742, 2012.

Michael Shulman.
Framed bicategories and monoidal fibrations.
20(18):80, 2007.

Thank you!