Orbit Class and its Applications

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Definition (Lusternik Schnirelmann category)

LS-category (category) of a topological space X

$$cat(X) = n$$

if n is the least integer such that there exists an open covering

$$X = \bigcup_{i=1}^{n} U_i$$

where each U_i is contractible to a point in X.

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If no such a covering exists we write $cat(X) = \infty$.

Group Action

- G: Topological group
- X: Hausdorff topological space
- $\Theta : G \times X \longrightarrow X$ is a map s.t.
  1. $\Theta(g, \Theta(h, x)) = \Theta(gh, x), \qquad \forall g, h \in G, x \in X;$
  2. $\Theta(e, x) = x, \quad \forall x \in X,\ e = 1_G.$

Θ is called an **action** of G on X, and X is called a G-**space**.

Definition

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- **Isotropy subgroup** of G at x:

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- **Orbit space**, $\mathbf{X/G}$: Set of all equivalence classes of orbits, endowed with the quotient topology.

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- Let U be a G-invariant subset of X,

$$H : U \times I \to X$$

is called a **G-homotopy** if

$$gH(x,t) = H(gx,t) \qquad \forall g \in G,\ x \in U,\ t \in I.$$

Definition

G-contractible Let U and A be G-invariant subsets of X. We say U is **G-contractible to A** and denote by

$$U \underset{\mathsf{G}}{\widetilde{\trianglerighteq}} A,$$

if there exists a G-homotopy $H : U \times I \to X$ s.t:

- H_0 is the inclusion of U in X.
- $H_1(U) \subseteq A$

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In this case we usually write,

$$H : U \underset{\mathsf{G}}{\widetilde{\trianglerighteq}} A.$$

Definition

X: A G-space

A G-invariant open subset U of X is called **G-categorical** if there exists an orbit $\mathcal{O}(x)$ such that $U \underset{G}{\widetilde{\trianglerighteq}} \mathcal{O}(x)$.

Definition

Equivariant LS-category of a topological space X

$$cat_G(X) = n$$

if n is the least integer such that there exists an open covering

$$X = \bigcup_{i=1}^{n} U_i$$

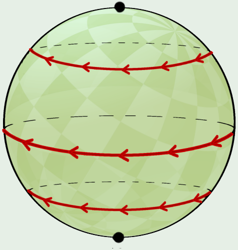
where each U_i is a G-categorical subset.

If no such a covering exists we write $cat_G(X) = \infty$.

Introduction

Example

$$\begin{array}{ccc} \mathbb{S}^1 \times \mathbb{S}^2 & \longrightarrow & \mathbb{S}^2 \\ t.(z,x) & \longmapsto & (t.z,x) \end{array}$$



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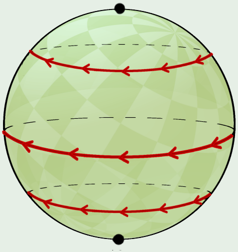

$$cat_{\mathbb{S}^1}(\mathbb{S}^2) = 2.$$

Introduction

Theorem (S. Hurder and D. Töben, 2015)

Let G be a Lie group acting properly on M. Let $\mathfrak{M}_0$ be the set of all locally minimal strata. Then

$$\sum_{\mathcal{M}_x \in \mathfrak{M}_0} cat_G(\mathcal{M}_x) \leq cat_G(M)$$

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- $M^x_{(H)}$: the connected component of $M_{(H)}$ containing x
- The conjugacy class (G_x) is called the orbit type of x
- $\mathcal{M}_x = \mathcal{O}\left(M^x_{(G_x)}\right) = G.\left(M^x_{(G_x)}\right)$

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In this case $\mathcal{M}_x$ is called a locally minimal stratum.

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- $\mathcal{O}(x) \sim \mathcal{O}(y)$ iff

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- We denote the equivalence class of $\mathcal{O}(x)$ by $[\mathcal{O}(x)]$ and call it the **orbit class** correspond to x.
- $$[\mathcal{O}(y)] \leq [\mathcal{O}(x)] \quad \text{iff} \quad \mathcal{O}(x) \underset{G}{\widetilde{\trianglerighteq}} \mathcal{O}(y).$$
- We call the Hasse diagram corresponding to this poset, an **orbit diagram** of X and denote by $\mathcal{OD}(G \curvearrowright X)$.

Example

Consider $\mathbb{S}^1$-action on $\mathbb{S}^2$.

$$\mathbb{S}^2 = \left\{ (z, x) \in \mathbb{C} \times \mathbb{R} \;\middle|\; |z|^2 + x^2 = 1 \right\}$$

$$t \cdot (z, x) = (tz, x)$$

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Orbit Classes:

- $\alpha_1 = \left[\mathcal{O}(0,1)\right], \quad \alpha_2 = \left[\mathcal{O}(0,-1)\right].$
- $\beta = \left[\mathcal{O}(1,0)\right].$

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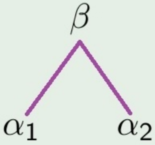
Orbit Diagram:



$\mathcal{OD}(\mathbb{S}^1 \curvearrowright \mathbb{S}^2)$

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Consider $\mathbb{T}^2$-action on $\mathbb{CP}^2$.

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Consider $\mathbb{T}^2$-action on $\mathbb{CP}^2$.

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Orbit Classes:

- $\alpha_1 = \Big[\mathcal{O}\big([z_1 : 0 : 0]\big)\Big]$, $\alpha_2 = \Big[\mathcal{O}\big([0 : z_2 : 0]\big)\Big]$,
  $\alpha_3 = \Big[\mathcal{O}\big([0 : 0 : z_3]\big)\Big]$.
- $\beta_1 = \Big[\mathcal{O}\big([z_1 : z_2 : 0]\big)\Big]$, $\beta_2 = \Big[\mathcal{O}\big([z_1 : 0 : z_3)\Big]$,
  $\beta_3 = \Big[\mathcal{O}\big([0 : z_2 : z_3]\big)\Big]$.
- $\delta = \Big[\mathcal{O}\big([z_1 : z_2 : z_3]\big)\Big]$

where $z_1, z_2, z_3 \in \mathbb{C} - \{0\}$.

Example

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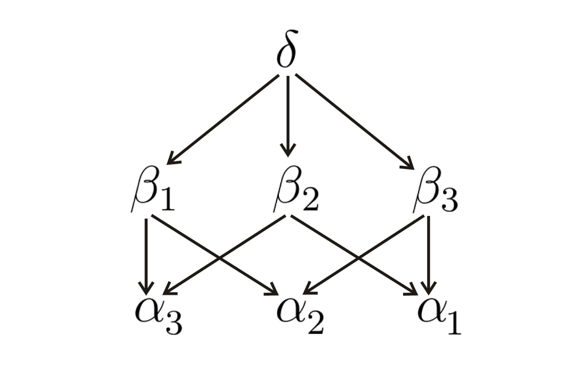

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- $\mathcal{O}(m)$ is called **minimal orbit** if $[\mathcal{O}(m)]$ is minimal with respect to $\leq$.
- All minimal orbit classes of X are placed at the bottom of $\mathcal{OD}(G \curvearrowright X)$.
- Any G-homotopy preserves the minimal orbits.

Theorem (M. Bayeh and S. Sarkar, 2015)

Let X be a G-space and $\Big[\mathcal{O}(x_i)\Big]_{i\in\mathcal{A}}$ be the minimal orbit classes in X,

$$X_i = \bigcup_{\mathcal{O}(y)\in[\mathcal{O}(x_i)]} \mathcal{O}(y)$$

Then

$$\sum_{i\in\mathcal{A}} cat_G(X_i) \leq cat_G(X).$$

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Then

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In particular

$$\#\mathcal{A} \leq \sum_{i\in\mathcal{A}} cat_G(X_i) \leq cat_G(X),$$

where $\#\mathcal{A}$ is the cardinal of $\mathcal{A}$.

Definition (G-connected Space)

A G-space X is called G-connected if for any closed subgroup $H \leq G$, X^H is path-connected.

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Theorem (M. Bayeh and S. Sarkar, 2015)

If G is path-connected and X has a unique minimal orbit class, then for any subgroup H of G, X^H is path-connected. In particular X is G-connected.

Orbit Class is very useful!!

Orbit Class

Definition

- stratification of a topological space X is a finite filtration by closed subsets X_i, where $X_i \setminus X_{i-1}$ is empty or a smooth submanifold of dimension i. Connected components of $X_i \setminus X_{i-1}$ are strata.

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- Whitney's condition A: U and W satisfy this condition if considering a sequence $x_1, x_2, \cdots$ in U converges to y in W, s.t.
  the sequence of tangent planes T_j to U at x_j converges to T,

  then T contains the tangent plane to W at y.

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- Whitney's condition B: U and W satisfy this condition if considering a sequence $x_1, x_2, \cdots$ in U and a sequence $y_1, y_2, \cdots$ in W both converge to y in W s.t.

  the secant lines $L_j = (x_j y_j)$ converges to a line L
  the sequence of tangent planes T_j to U at x_j converges to T

  then T contains L.

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Theorem

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Definition

Let X be a G-space. The orbit class category of, denoted by $\mathcal{O}_{\mathbf{G}}(\mathbf{X})$, is a category whose

- objects are orbits $\mathcal{O}(x)$,
- morphisms are G-contractible maps; i.e.

$$\mathcal{O}(x) \to \mathcal{O}(y) \iff \mathcal{O}(x) \underset{G}{\widetilde{\trianglerighteq}} \mathcal{O}(y)$$

Orbit Class Category

There is a functor $\mathfrak{G}$ from the orbit class category $\mathcal{O}_{\mathbf{G}}(\mathbf{X})$ to the orbit category $\mathcal{O}^*_{\mathbf{G}}$.

$$\begin{aligned} \mathfrak{G} : \mathcal{O}_{\mathbf{G}}(\mathbf{X}) &\longrightarrow \mathcal{O}^*_{\mathbf{G}} \\ \mathcal{O}(x) &\mapsto (G/G_x) \\ \mathcal{O}(x) \to \mathcal{O}(y) &\mapsto (G/G_x) \to (G/G_y) \end{aligned}$$

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If X is G-connected, then the functor $\mathfrak{G}$ is full.

Definition

Let $F : \mathfrak{C} \to \mathfrak{D}$ be a functor.
A morphism $f : c_1 \to c_0$ in $\mathfrak{C}$ is **cartesian** iff for any morphism $g : c_2 \to c_0$ in $\mathfrak{C}$ and $\varphi : F(c_2) \to F(c_1)$ in $\mathfrak{D}$ s.t.

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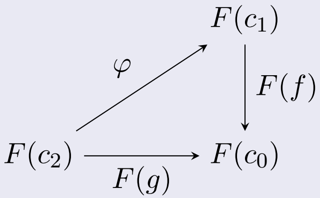

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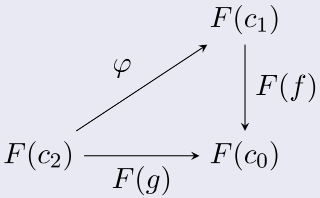

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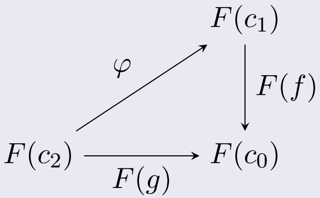

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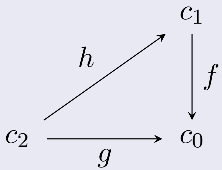

Grothendieck fibration

A functor $F : \mathfrak{C} \to \mathfrak{D}$ is called **Grothendieck fibration** if for every object c_0 of $\mathfrak{C}$ and morphism $\alpha : d \to F(c_0)$ in $\mathfrak{D}$, there exists a cartesian arrow $f : c_1 \to c_0$ with $F(f) = \alpha$.

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Theorem

Let X be a G-connected space. Then the functor

$$\mathfrak{G} : \mathcal{O}_{\mathbf{G}}(\mathbf{X}) \to \mathcal{O}^*_{\mathbf{G}}$$

is a Grothendieck fibration.

Definition

Let X and Y be two G-spaces. A map $\varphi : X \to Y$ is called special G-map if for every $g \in G$ and $x \in X$,

- $\varphi(gx) = g\varphi(x)$
- $G_{\varphi(x)} = G_x$

Orbit Class Category

Theorem

Every special G-map $\varphi : X \to Y$ induces a functor

$$\varphi^* : \mathcal{O}_G(X) \to \mathcal{O}_G(Y)$$

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Every special G-map $\varphi : X \to Y$ induces a functor

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such that

$$\begin{array}{ccc} \mathcal{O}_G(X) & \xrightarrow{\quad\varphi^*\quad} & \mathcal{O}_G(Y) \\ & {\scriptstyle \mathfrak{G}_X}\searrow \qquad \swarrow{\scriptstyle \mathfrak{G}_Y} & \\ & \mathcal{O}_G^* & \end{array}$$

Theorem

Let $\varphi, \psi : X \to Y$ be two special G-maps. If φ and ψ are G-homotopic, then there exists a natural transformation (natural isomorphism)

$$\eta : \varphi^* \Rightarrow \psi^*$$

Thank you!

In memory of

Maryam Mirzakhani

(the first woman who was awarded the Fields Medal)

"The beauty of mathematics only shows itself to more patient followers"