Enriched algebraic weak factorisation systems

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Enriched (co)fibrant replacement

Theorem (Folklore: Garner, Riehl, Shulman, ...)

Let $\mathcal{V}$ be a monoidal model category in which every object is cofibrant. Then any cofibrantly generated model $\mathcal{V}$-category has:

- *a cofibrant replacement $\mathcal{V}$-comonad, and*
- *a fibrant replacement $\mathcal{V}$-monad.*

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Theorem (Lack–Rosický)

Let $\mathcal{V}$ be a monoidal model category with cofibrant unit object. If $\mathcal{V}$ has a cofibrant replacement $\mathcal{V}$-comonad, then every object of $\mathcal{V}$ is cofibrant.

The problem of enriched (co)fibrant generation

Question

If $\mathcal{V}$ is a monoidal model category in which not every object is cofibrant, then what extra structure, if not an enrichment in the ordinary sense, is naturally possessed by the (co)fibrant replacement (co)monad of a model $\mathcal{V}$-category?

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An analysis of the monoidal model category $\mathcal{V} =$ **2-Cat** suggests the decisive concept:

$$\boxed{\textbf{locally weak } \mathcal{V}\textbf{-functor}}$$

Outline

1. The problem of enriched (co)fibrant replacement
2. The monoidal model category of 2-categories
3. Locally weak $\mathcal{V}$-functors
4. Monoidal and enriched AWFS

Outline

The monoidal category of 2-categories

Recall (Gray)

The category **2-Cat** of (small) 2-categories and 2-functors is a symmetric monoidal closed category with:

- unit object 1,
- tensor product $A \otimes B$ the (pseudo) Gray tensor product of 2-categories,
- internal hom $\mathbf{Gray}(A, B)$ the 2-category of 2-functors $A \longrightarrow B$, pseudonatural transformations, and modifications.

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Categories enriched over this monoidal category are called **Gray**-categories. The self-enrichment of **2-Cat** is called **Gray**.

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A 2-category is cofibrant if and only if its underlying category is free on a graph. In particular the unit 2-category 1 is cofibrant.

Since not every 2-category is cofibrant, it follows from the argument of Lack and Rosický that there does not exist a **Gray**-enriched cofibrant replacement comonad on **2-Cat**.

The strictification adjunction

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The model category **2-Cat** has a canonical cofibrant replacement comonad, which is induced by the adjunction

$$\mathbf{2\text{-}Cat} \underset{\longrightarrow}{\overset{\mathbf{st}}{\underset{\perp}{\longleftarrow}}} \mathbf{Bicat}$$

where **Bicat** is the category of bicategories and pseudofunctors, the right adjoint is the inclusion, and the left adjoint **st** sends a bicategory to its "strictification".

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Hence this canonical cofibrant replacement $\mathbf{st}A$ of a 2-category A is its "pseudofunctor classifier"; i.e. it has the universal property:

$$\frac{\mathbf{st}A \longrightarrow B \qquad \text{2-functors}}{A \rightsquigarrow B \qquad \text{pseudofunctors}}$$

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Theorem (C.)

The strictification adjunction extends to an adjunction of multicategories, i.e. an adjunction in the 2-category of multicategories.

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Hence the comonad **st** on **2-Cat** extends to a comonad in the 2-category of multicategories. But the multicategory structure on **2-Cat** is representable, so **st** in fact extends to a monoidal comonad on **2-Cat**.

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which, by the universal property of the pseudofunctor classifier, are equally pseudofunctors

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Corollary (C.)

The strictification comonad **st** *is a locally weak* **Gray***-comonad on* **Gray**.

Outline

Locally weak $\mathcal{V}$-functors

Let $(Q, \varphi, \varphi_0, \ldots)$ be a monoidal comonad on a monoidal category $\mathcal{V}$. We think of morphisms $QX \longrightarrow Y$ in $\mathcal{V}$ as "weak morphisms" $X \rightsquigarrow Y$ in $\mathcal{V}$.

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Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories. A *locally Q-weak $\mathcal{V}$-functor* $F \colon \mathcal{A} \longrightarrow \mathcal{B}$ consists of:

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(ii) for each $A, B \in \mathcal{A}$, a morphism $\psi_{A,B} \colon Q\mathcal{A}(A, B) \longrightarrow \mathcal{B}(FA, FB)$ in $\mathcal{V}$, i.e. a "weak morphism" $\psi_{A,B} \colon \mathcal{A}(A, B) \rightsquigarrow \mathcal{B}(FA, FB)$ in $\mathcal{V}$, Let $(Q, \varphi, \varphi_0, \ldots)$ be a monoidal comonad on a monoidal category $\mathcal{V}$. We think of morphisms $QX \longrightarrow Y$ in $\mathcal{V}$ as "weak morphisms" $X \rightsquigarrow Y$ in $\mathcal{V}$.

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subject to the following two axioms.

$$\begin{array}{ccccc} Q\mathcal{A}(B,C) \otimes Q\mathcal{A}(A,B) & \xrightarrow{\varphi} & Q\left(\mathcal{A}(B,C) \otimes \mathcal{A}(A,B)\right) & \xrightarrow{QK} & Q\mathcal{A}(A,C) \\ \downarrow{\scriptstyle \psi \otimes \psi} & & & & \downarrow{\scriptstyle \psi} \\ \mathcal{B}(FB,FC) \otimes \mathcal{B}(FA,FB) & & \xrightarrow[K]{\hspace{6em}} & & \mathcal{B}(FA,FC) \end{array}$$

$$\begin{array}{ccc} QI & \xrightarrow{Qj} & Q\mathcal{A}(A,A) \\ {\scriptstyle \varphi_0}\uparrow & & \downarrow{\scriptstyle \psi} \\ I & \xrightarrow[j]{} & \mathcal{B}(FA,FA) \end{array}$$

The Kleisli 2-category of locally weak $\mathcal{V}$-functors

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A (co)monad in this 2-category is called a *locally Q-weak $\mathcal{V}$-(co)monad*.

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Leibniz–Day constructions I

Let $(\mathcal{V}, \otimes, I)$ be a monoidal category with finite colimits and finite limits, such that $\otimes$ preserves finite colimits in each variable.

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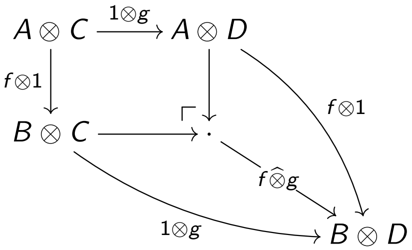

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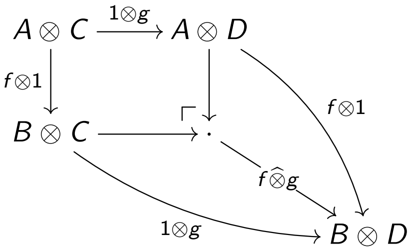

- definition of the associativity and unit constraints requires the above assumption on colimits.

Let $\mathcal{A}$ be a $\mathcal{V}$-category. Then $\mathcal{A}^{\mathbf{2}}$ is a $\mathcal{V}^{\mathbf{2}}$-category, with homs $\widehat{\mathcal{A}(f,g)}$ given by:

Leibniz–Day constructions II

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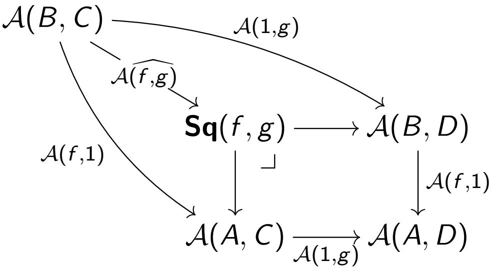

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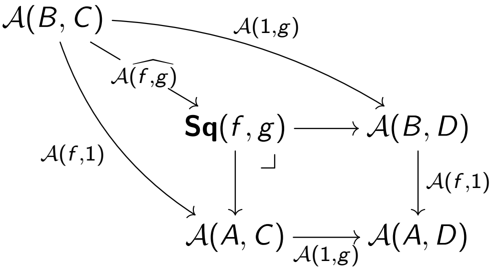
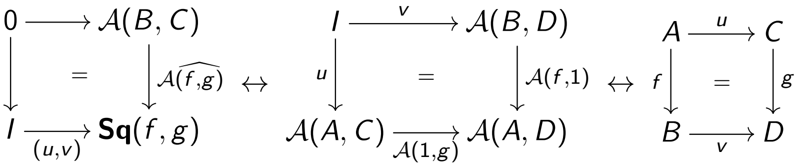


$\mathbf{Sq}(f,g)$ is the $\mathcal{V}$-object of squares $f \to g$.



Weak factorisation systems

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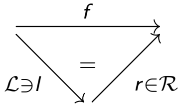

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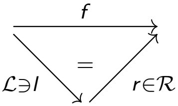
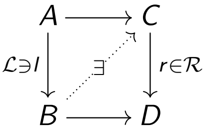


(ii) every square $l \to r$ has a diagonal filler:



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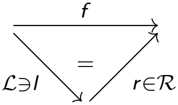

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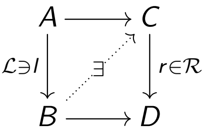

i.e. $$\mathcal{C}(B, C) \xrightarrow{\widehat{\mathcal{C}(l,r)}} \mathbf{Sq}(l, r)$$ is surjective $\forall l \in \mathcal{L}, r \in \mathcal{R}$.

Let $(\mathcal{L}, \mathcal{R})$ be a WFS on a monoidal category $\mathcal{V}$. A WFS $(\mathcal{H}, \mathcal{M})$ on a $\mathcal{V}$-category $\mathcal{A}$ is said to be *enriched over* $(\mathcal{L}, \mathcal{R})$ if for each $A \xrightarrow{f} B$ in $\mathcal{H}$ and each $C \xrightarrow{g} D$ in $\mathcal{M}$, the morphism $\mathcal{A}(B, C) \xrightarrow{\widehat{\mathcal{A}(f,g)}} \mathbf{Sq}(f, g)$ in $\mathcal{V}$ belongs to $\mathcal{R}$.

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Examples

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(b) A WFS enriched over the (all, iso) factorisation system on **Set** is precisely an orthogonal factorisation system.

(c) Let $\mathcal{V}$ be a monoidal model category. The two defining WFS of a model $\mathcal{V}$-category are enriched over the (cofibration, trivial fibration) WFS on $\mathcal{V}$.

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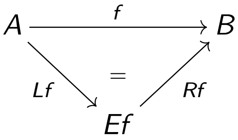

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$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle Lf}\searrow & = & \nearrow{\scriptstyle Rf} \\ & Ef & \end{array}$$

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"L-map" $\equiv$ L-coalgebra

$$\begin{array}{ccc} A & \xrightarrow{Lf} & Ef \\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle s} & \downarrow{\scriptstyle Rf} \\ B & \xrightarrow[1]{} & B \end{array}$$

"R-map" $\equiv$ R-algebra

$$\begin{array}{ccc} C & \xrightarrow{1} & C \\ {\scriptstyle Lg}\downarrow & \nearrow{\scriptstyle p} & \downarrow{\scriptstyle g} \\ Eg & \xrightarrow[Rg]{} & D \end{array}$$

Algebraic weak factorisation systems

Each square in $\mathcal{C}$ from an L-coalgebra (f, s) to an R-algebra (g, p)

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & = & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D \end{array}$$

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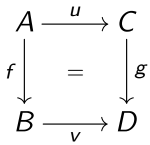
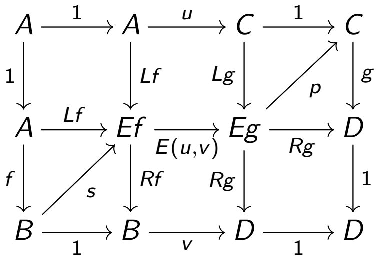


has the canonical diagonal filler $p \circ E(u, v) \circ s$.



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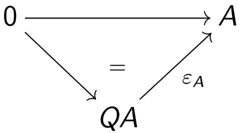
If (L, R) is an AWFS on a category $\mathcal{C}$ with an initial object 0, then factorisation of morphisms of the form



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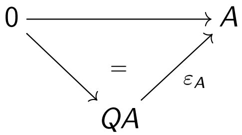

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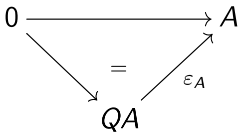

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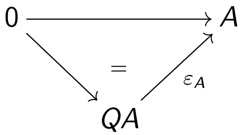

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Dually, if $\mathcal{C}$ has a terminal object 1, then factorisation of morphisms of the form $A \longrightarrow 1$ defines a monad on $\mathcal{C}$, called the fibrant replacement monad.

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Two-variable oplax morphism of AWFS

Axiom (iii) ($\otimes$ is a two-variable oplax morphism of AWFS) implies, inter alia, the following result.

Proposition (Riehl)

The tensor product $\widehat{\otimes}$ on $\mathcal{V}^{\mathbf{2}}$ lifts to a functor

$$\widehat{\otimes}\colon L\text{-Coalg} \times L\text{-Coalg} \longrightarrow L\text{-Coalg}.$$

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Moreover, by the definition of two-variable oplax morphisms of AWFS, φ defines natural transformations

$$Lf \widehat{\otimes} Lg \xrightarrow{\Phi} L(f \widehat{\otimes} g) \qquad Lf \widehat{\otimes} Rg \xrightarrow{\Sigma} R(f \widehat{\otimes} g) \qquad Rf \widehat{\otimes} Lg \xrightarrow{\Pi} R(f \widehat{\otimes} g)$$

which, together with the remaining axioms, prove the following theorem.

Cofibrant replacement is a monoidal comonad

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Corollary

(i) *The monoidal structure on $\mathcal{V}^{\mathbf{2}}$ lifts to a monoidal structure on $L\text{-}\mathrm{Coalg}$.*
(ii) *$R\text{-}\mathrm{Kl}$ is a two-sided $(L\text{-}\mathrm{Coalg})$-actegory.*
(iii) *The monoidal structure on $\mathcal{V}$ lifts to a monoidal structure on $Q\text{-}\mathrm{Coalg}$.*
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Let (L, R) be a monoidal AWFS on $\mathcal{V}$ with cofibrant replacement comonad Q. Recall that the Kleisli category $\mathcal{V}_Q$ for Q is called the category of weak maps for (L, R).

The multicategory of weak maps

Let (L, R) be a monoidal AWFS on $\mathcal{V}$ with cofibrant replacement comonad Q. Recall that the Kleisli category $\mathcal{V}_Q$ for Q is called the category of weak maps for (L, R).

Corollary

The Kleisli adjunction for Q extends to an adjunction of multicategories.

$$\mathcal{V} \underset{\longrightarrow}{\overset{Q}{\underset{\perp}{\longleftarrow}}} \mathcal{V}_Q$$

n-ary morphisms $(X_1, \ldots, X_n) \longrightarrow Y$ in the multicategory structure on $\mathcal{V}_Q$ are morphisms $QX_1 \otimes \cdots \otimes QX_n \longrightarrow Y$ in $\mathcal{V}$.

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$$\begin{array}{ccccc} E\widehat{\mathcal{A}(g,h)} \otimes E\widehat{\mathcal{A}(f,g)} & \xrightarrow{\varphi} & E\left(\widehat{\mathcal{A}(g,h)}\widehat{\otimes}\widehat{\mathcal{A}(f,g)}\right) & \xrightarrow{E\widehat{K}} & E\widehat{\mathcal{A}(f,h)} \\ {\scriptstyle \psi\otimes\psi}\downarrow & & & & \downarrow{\scriptstyle \psi} \\ \mathcal{A}(Ng,Nh) \otimes \mathcal{A}(Nf,Ng) & & \xrightarrow[K]{\qquad\qquad} & & \mathcal{A}(Nf,Nh) \end{array} \qquad \begin{array}{ccc} QI & \xrightarrow{E\widehat{j}} & E\widehat{\mathcal{A}(f,f)} \\ {\scriptstyle \varphi_0}\uparrow & & \downarrow{\scriptstyle \psi} \\ I & \xrightarrow[j]{} & \mathcal{A}(Nf,Nf) \end{array}$$

Axiom (ii) ($\mathcal{A}(-,-)$ is a two-variable lax morphism of AWFS) implies, inter alia, the following result.

Proposition (Riehl)

The $\mathcal{V}^2$-valued hom $\mathcal{A}\widehat{(-,-)}$ on $\mathcal{A}^2$ lifts to a functor

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Moreover, by the definition of two-variable lax morphisms of AWFS, ψ defines natural transformations

$$R\mathcal{A}\widehat{(f,g)} \xrightarrow{\Theta} \mathcal{A}\widehat{(Hf,Mg)}$$

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Let (H, M) be an (L, R)-enriched AWFS *on $\mathcal{A}$. Then the following are true.*

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Let (L, R) be a monoidal AWFS on $\mathcal{V}$ with cofibrant replacement comonad Q. Let (H, M) be a (L, R)-enriched AWFS on a $\mathcal{V}$-category $\mathcal{A}$ with cofibrant replacement comonad S.

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Corollary (C.)

*The Kleisli adjunction for S extends to a $\mathcal{V}_Q$-enriched adjunction, i.e. an adjunction in the 2-category $\mathcal{V}_Q$-**Cat** of categories enriched over the multicategory of weak maps for (L, R).*

$$\mathcal{A} \underset{\longrightarrow}{\overset{S}{\underset{\perp}{\longleftarrow}}} \mathcal{A}_S$$

The hom-objects in the $\mathcal{V}_Q$-category $\mathcal{A}_S$ are $\mathcal{A}_S(A, B) = \mathcal{A}(SA, B)$.

Examples of monoidal and enriched AWFS

Examples

(a) Every monoidal AWFS on a monoidal closed category is enriched over itself.

(b) The (all,iso) factorisation system on a monoidal category $\mathcal{V}$ is a monoidal AWFS (with canonical factorisation $f = 1 \circ f$) . An AWFS on a $\mathcal{V}$-category $\mathcal{A}$ enriched over this monoidal AWFS is precisely a $\mathcal{V}$-enriched orthogonal factorisation system on $\mathcal{A}$.

(c) The "split epi" AWFS on **Set** (in which $f \colon X \to Y$ factors through $X + Y$) is monoidal with respect to cartesian product. Every AWFS is canonically enriched over this monoidal AWFS.

(d) Let $\mathcal{V}$ be a monoidally cocomplete category, so that $U = \mathcal{V}(I, -) \colon \mathcal{V} \to \mathbf{Set}$ has a left adjoint F. The "U-split epi" AWFS on $\mathcal{V}$ (in which $f \colon X \to Y$ factors through $X + FUY$) is monoidal. Every AWFS on a $\mathcal{V}$-category is canonically enriched over this monoidal AWFS.