A symmetric monoidal and compact closed bicategorical syntax for graphical calculi

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21 July 2017

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motivation

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a question

Is there a general framework for systems comprised of open networks and rewriting?

Loosely, by *open network* we mean a graphical language with inputs and outputs

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Is there a general framework for systems comprised of open networks and rewriting?

Loosely, by *open network* we mean a graphical language with inputs and outputs

goals

Today, we will

construct such a bicategorical framework

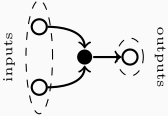
— and —

illustrate its use on the zx-calculus

modeling open networks & rewrites

modeling open networks

Open networks can be modeled with cospans, eg



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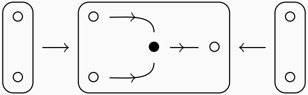

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$$X \to G \leftarrow Y$$

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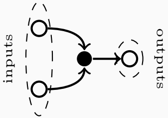

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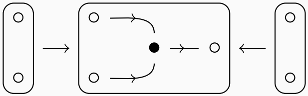

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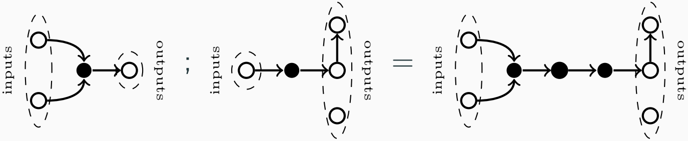
Compatible open networks can be connected, e.g.



This is made precise with pushouts:

$$(X \to G \leftarrow Y);(Y \to H \leftarrow Z) = (X \to G +_Y H \leftarrow Z)$$

This induces a category with

(objects)	input and output types
(morphisms)	open networks possibly modulo relations.

Can we categorify this with relations as 2-cells?

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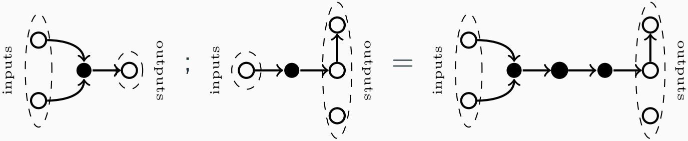

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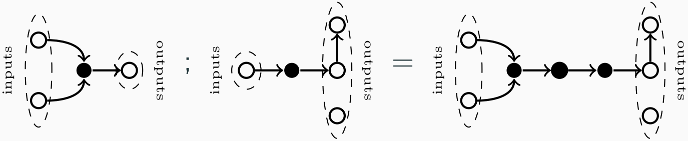

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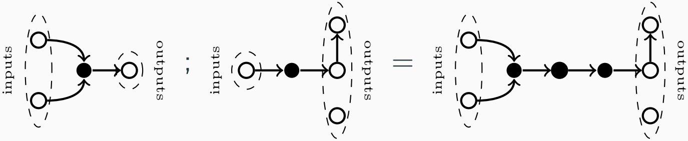

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modeling rewrite rules

Using graph-like structures, we give relations by rewrite rules.

In particular, we use **double pushout rewriting** where a rule

$$L \rightsquigarrow R$$

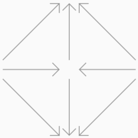
is given by a span

$$L \leftarrow K \rightarrow R$$

So what we want is

rewrite rules (spans) between **open networks** (cospans).

Thus **spans of cospans**:



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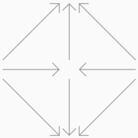

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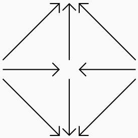

combining open networks & rewrite rules

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The components we are working with are

- inputs and outputs
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combining open networks & rewrite rules

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Did we just describe a bicategory?

combining open networks & rewrite rules

Theorem (C.)

Let $\mathbf{T}$ *be a topos. There is a bicategory* $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ *with*

(0-cells) *objects of* $\mathbf{T}$
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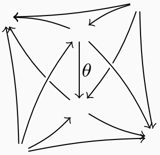

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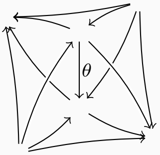

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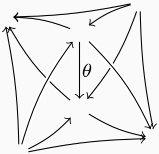

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combining open networks & rewrite rules

In case monic span legs are too strict...

Theorem (C.)
Let C be a category with finite limits and colimits. There is a bicategory $\mathbf{Sp}(\mathbf{Csp}(\mathbf{C}))$ *with*

(0-cells) *objects of* $\mathbf{C}$,
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combining open networks & rewrite rules

Theorem (C. & Courser)

Consider the topos T and the finitely complete and cocomplete category C to be symmetric monoical via $+$ and 0.

Then the bicategories $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ *and* $\mathbf{Sp}(\mathbf{Csp}(\mathbf{C}))$ *are symmetric monoidal and compact closed (á la Mike Stay).*

$\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ and $\mathbf{Sp}(\mathbf{Csp}(\mathbf{C}))$ are too big!

We need to pare them down

Let's illustrate this process with the zx-calculus

combining open networks & rewrite rules

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Let's illustrate this process with the zx-calculus

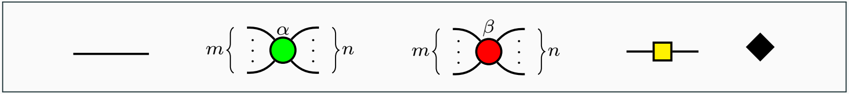
the zx-calculus

the zx-calculus – generators

The zx-calculus[1] is a syntax used in categorical quantum mechanics.

It models certain quantum processes

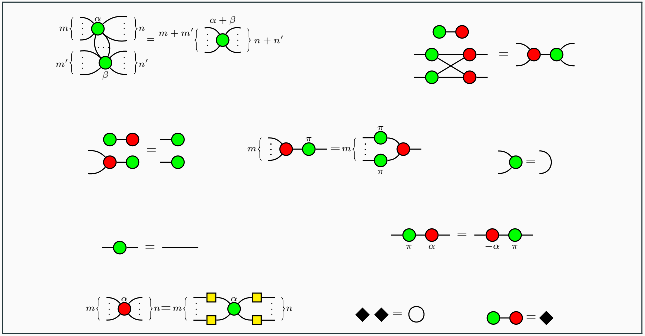
It is generated by the diagrams



[1] B Coecke & R Duncan (2011) *Interacting quantum observables: categorical algebra and diagrammatics*. New J. Phys., 13 (4), 043016.

the zx-calculus – generators

and the relations



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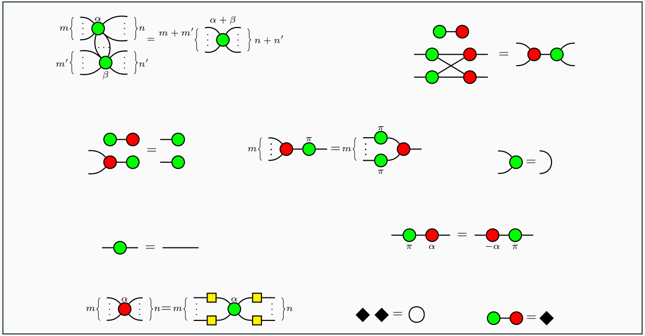

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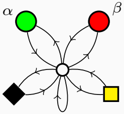
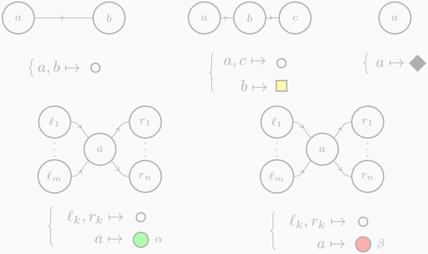
the zx-calculus – coloring the nodes

We want directed graphs with colored nodes. To this end, we define a graph S_{zx}



$\alpha, \beta \in [-\pi, \pi)$

The generating zx-diagrams are almost **graphs over** S_{zx}



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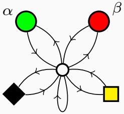

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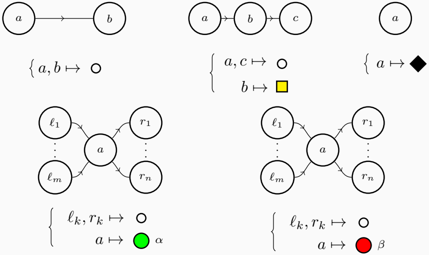

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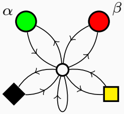

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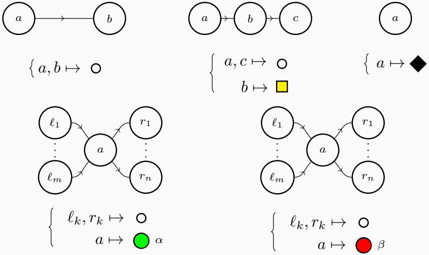

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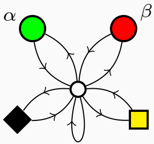
Define a functor

$$N\colon \mathbf{FinSet} \to \mathbf{Graph} \downarrow S_{\mathsf{zx}}$$

by sending a set x to the edgeless graph with node set x equipped with the map constant over the node ○ of



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the zx-calculus – constructing inputs and outputs

Rewrite is the SMCC sub-bicategory of $\mathbf{Sp}(\mathbf{Csp}(\mathbf{Graph} \downarrow S_{zx}))$

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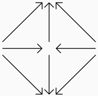
(2-cells)		all DPO rewrite rules

Rewrite is still too big. <u>WHAT IS IT GOOD FOR?</u>

– an ambient space in which to generate SMCC bicategories –

To categorify the zx-calculus, we will translate

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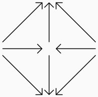
(2-cells)		all DPO rewrite rules

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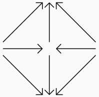
(2-cells)		all DPO rewrite rules

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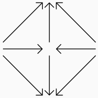
(2-cells)		all DPO rewrite rules

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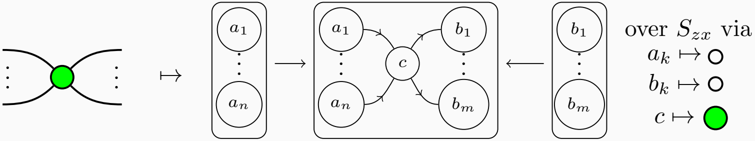
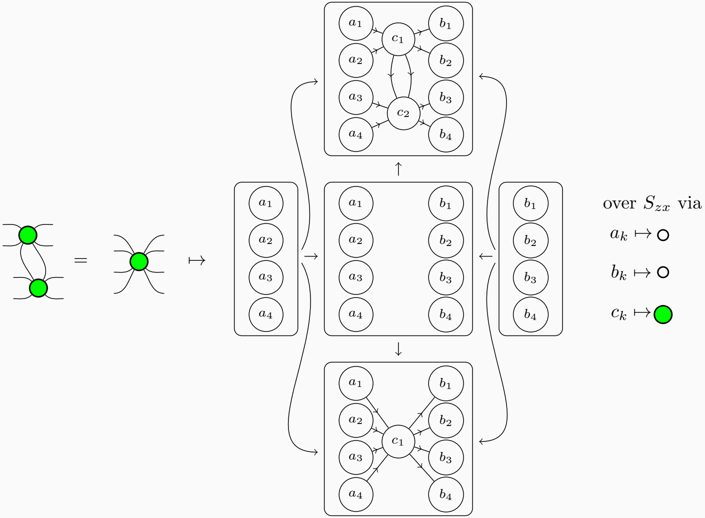
Translate zx-diagrams into 1-cells of **Rewrite**



etc.

the zx-calculus – translating to Rewrite

Translate zx-relations into 2-cells of **Rewrite**



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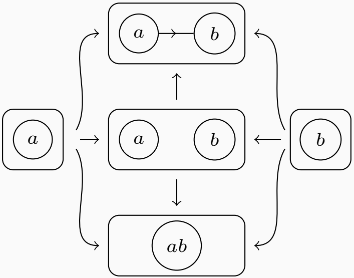
To force the wire to act like the identity, we add the 2-cell



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$a \mapsto$ ○

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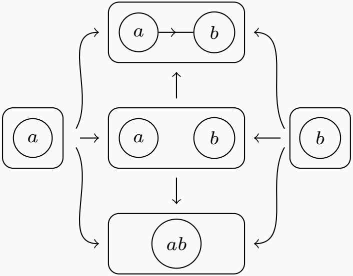

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the zx-calculus – a bicategory

Denote by **zx** the category with

(objects) $\mathbb{N}$

(morphisms) zx-diagrams modulo zx-relations

Theorem (C.)

Let $||\underline{\mathbf{zx}}||$ *be the category with*

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Then $||\underline{\mathbf{zx}}||$ *is equivalent to* **zx**

This equivalence is witnessed by the functor described in the above translation process.

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