A relative monotone-light factorization system for internal groupoids

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(joint work with T. Everaert and M. Gran)

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Aim of the talk

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[Bourn '87]

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Aim: clarify connections between the two, with a special attention to the Mal'tsev case.

Internal groupoids

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$$\begin{tikzcd} X_1 \times_{(d,c)} X_1 \arrow[r, shift left=2, "pr_2"] \arrow[r, "m" description] \arrow[r, shift right=2, "pr_1"'] & X_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0 \arrow[l, "e" description] \end{tikzcd}$$

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A functor $F\colon \mathbb{X} \to \mathbb{Y}$ between internal groupoids is given by a pair of morphisms (f_0, f_1)

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X_1 \arrow[r, "f_1"] \arrow[d, shift right=2, "c"'] \arrow[d, shift left=2, "d"] & Y_1 \arrow[d, shift right=2, "c"'] \arrow[d, shift left=2, "d"] \\
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$$D(X) = \begin{tikzcd} X \arrow[r, shift left=2, "1"] \arrow[r, "1" description] \arrow[r, shift right=2, "1"'] & X \arrow[loop above, "1"] \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "1"'] & X \arrow[l, "1" description] \end{tikzcd}$$

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Supp is defined via the following factorization

$$\begin{tikzcd}
X_1 \arrow[r, two heads] \arrow[rr, bend left, "\langle c, d \rangle"] \arrow[d, shift right=2ex, "c"'] \arrow[d, shift left=2ex, "d"] & (\mathrm{Supp}(\mathbb{X}))_1 \arrow[r, tail] \arrow[d, shift right=2ex] \arrow[d, shift left=2ex] & X_0 \times X_0 \arrow[d, shift right=2ex, "pr_1"'] \arrow[d, shift left=2ex, "pr_2"] \\
X_0 \arrow[u, "e" description] \arrow[r, equal] & X_0 \arrow[u] \arrow[r, equal] & X_0 \arrow[u, "\Delta" description]
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These are also called

- semi-localization [Mantovani '98]
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Under the conditions above, the reflection induces a factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{B}$

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Factorization of an arrow f

$$\begin{tikzcd}
A \arrow[dr, "e"'] \arrow[drr, bend left, "\eta_A"] \arrow[ddr, bend right, "f"'] & & \\
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$\mathcal{M}^*$ is the class of morphisms f for which there exists some effective descent morphism p such that $p^*(f)$ is in $\mathcal{M}$

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Examples: [Everaert, Gran '13]

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$\pi_1(\mathbb{X}) = \mathrm{Ker}(\langle d, c \rangle \colon X_1 \to X_0 \times X_0)$

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$$
\begin{tikzcd}
 & 0 \arrow[rr] \arrow[dl] \arrow[dd, shift right=2] \arrow[dd, leftarrow] \arrow[dd, shift left=2] & & A \arrow[dl, "{\langle 0,\langle 1_A,1_A\rangle\rangle}"'] \arrow[dd, shift right=2, "1_A"'] \arrow[dd, leftarrow] \arrow[dd, shift left=2, "1_A"] \\
A \arrow[rr, "{\langle 1_A,0\rangle}"] \arrow[dd, shift right=2, "1_A"'] \arrow[dd, leftarrow] \arrow[dd, shift left=2, "1_A"] & & A \times (A \times A) \arrow[dd, shift right=2, "1_A \times pr_1"'] \arrow[dd, leftarrow] \arrow[dd, shift left=2, "1_A \times pr_2"] & \\
 & 0 \arrow[rr] \arrow[dl] & & A \arrow[dl, "{\langle 0,1_A\rangle}"] \\
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Applying π_0 we have

$$
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A relevant case: [Janelidze, Kelly '94]

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is said to be admissible w.r.t. $\mathcal{F}$ if I preserves pullbacks of the form

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$(\mathcal{E}', \mathcal{M}^*)$ may also be a factorization system for $\mathcal{F}$.

Internal groupoids in Mal'tsev

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Proposition ([Gran '01])

A regular epimorphism F in $\mathbf{Gpd}(\mathcal{C})$ is

- *a trivial extension if and only if both F and $\mathrm{Supp}(F)$ are discrete fibrations;*
- *a central extension if and only if it is a discrete fibration.*

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Proposition

Final functors between internal groupoids in $\mathcal{C}$ are stable under pullback along regular epimorphisms. Equivalently, the pair

(final functors, reg. epic discrete fibrations)

is a factorization system for regular epimorphisms in $\mathbf{Gpd}(\mathcal{C})$.

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