

# A relative monotone-light factorization system for internal groupoids

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(joint work with T. Everaert and M. Gran)

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# Aim of the talk

[Bourn '87]

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Aim: clarify connections between the two, with a special attention to the Mal'tsev case.





# Internal groupoids

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A functor  $F: \mathbb{X} \rightarrow \mathbb{Y}$  between internal groupoids is given by a pair of morphisms  $(f_0, f_1)$

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 X_1 & \xrightarrow{f_1} & Y_1 \\
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$$D(X) = X \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{1} \\ \xrightarrow{1} \end{array} X \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \\ \xrightarrow{1} \end{array} X$$



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The adjunction  $\pi_0 \dashv D$  can be seen as a composite

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Supp is defined via the following factorization

$$\begin{array}{ccccc}
 & & \langle c, d \rangle & & \\
 & & \curvearrowright & & \\
 X_1 & \twoheadrightarrow & (\text{Supp}(\mathbb{X}))_1 & \twoheadrightarrow & X_0 \times X_0 \\
 \begin{array}{c} \uparrow \\ c \\ \downarrow \\ \uparrow \\ e \\ \downarrow \\ d \\ \downarrow \end{array} & & \begin{array}{c} \uparrow \\ \downarrow \\ \uparrow \\ \downarrow \end{array} & & \begin{array}{c} \uparrow \\ \text{pr}_1 \\ \downarrow \\ \uparrow \\ \text{pr}_2 \\ \downarrow \end{array} \\
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- ▶ semi-localization [Mantovani '98]
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Under the conditions above, the reflection induces a factorization system  $(\mathcal{E}, \mathcal{M})$  on  $\mathcal{B}$



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Factorization of an arrow  $f$

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$\mathcal{M}^*$  is the class of morphisms  $f$  for which there exists some effective descent morphism  $p$  such that  $p^*(f)$  is in  $\mathcal{M}$

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Examples: [Everaert, Gran '13]



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*A functor  $F$  between internal groupoids in a semi-abelian category is final if and only if  $\pi_0(F)$  is an iso and  $\pi_1(F)$  is a regular epi.*

## The case of $\pi_0$

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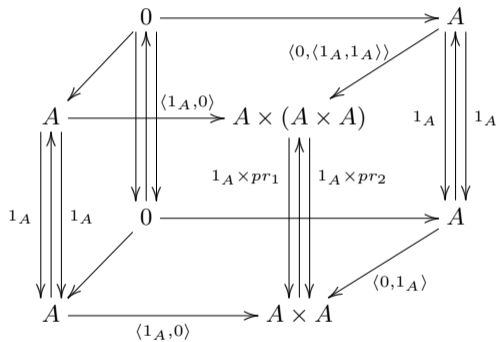
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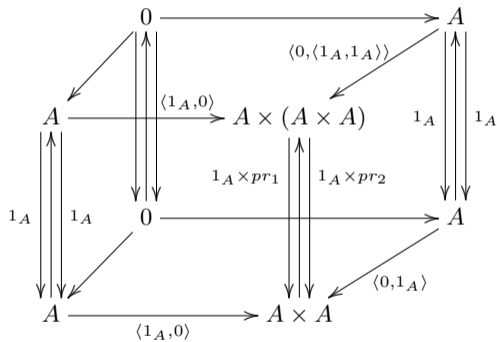
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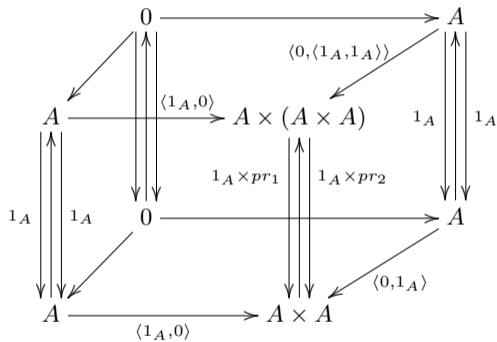


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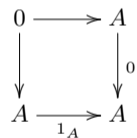
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# A counterexample



$\pi_1$ 's are trivial

Applying  $\pi_0$  we have





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$\mathcal{M}$  = trivial extensions,  $\mathcal{M}^*$  = central extensions.





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- ▶  $\mathcal{E}$  and  $\mathcal{M}$  both contain identities and are closed under composition with isomorphisms;
- ▶  $\mathcal{E}$  and  $\mathcal{M}$  are orthogonal to each other;
- ▶  $\mathcal{M}$  is contained in  $\mathcal{F}$ ;
- ▶ every arrow  $f$  in  $\mathcal{F}$  is the composite  $f = me$  of an  $m$  in  $\mathcal{M}$  with an  $e$  in  $\mathcal{E}$ .

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# Relative factorization systems

[Chikhladze '04]

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- ▶ every arrow  $f$  in  $\mathcal{F}$  is the composite  $f = me$  of an  $m$  in  $\mathcal{M}$  with an  $e$  in  $\mathcal{E}$ .

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$(\mathcal{E}, \mathcal{M})$  is a factorization system for  $\mathcal{F}$ .



$\mathcal{E}'$  the class of morphisms in  $\mathcal{B}$  whose pullbacks along arrows in  $\mathcal{F}$  are in  $\mathcal{E}$ .  
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$(\mathcal{E}', \mathcal{M}^*)$  may also be a factorization system for  $\mathcal{F}$ .





# Internal groupoids in Mal'tsev

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$\mathcal{C}$  is an admissible Birkhoff subcategory of  $\mathbf{Gpd}(\mathcal{C})$ :

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Proposition ([Gran '01])

A regular epimorphism  $F$  in  $\mathbf{Gpd}(\mathcal{C})$  is

- ▶ a trivial extension if and only if both  $F$  and  $\text{Supp}(F)$  are discrete fibrations;
- ▶ a central extension if and only if it is a discrete fibration.



$(\mathcal{E}, \mathcal{M}) = (\pi_0\text{-invertible, trivial extensions})$  form a factorization system for regular epimorphisms in  $\mathbf{Gpd}(\mathcal{C})$



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### Proposition

*Final functors between internal groupoids in  $\mathcal{C}$  are stable under pullback along regular epimorphisms. Equivalently, the pair*

*(final functors, reg. epic discrete fibrations)*

*is a factorization system for regular epimorphisms in  $\mathbf{Gpd}(\mathcal{C})$ .*

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