Differential equations in tangent categories

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Overview

- Up to now, only the *differential* side of differential geometry has been developed for tangent categories.
- One aspect of the *integral* side of differential geometry are integral curves, i.e., solutions to differential equations.
- In this talk, we'll see how to discuss differential equations and their solutions in a tangent category: this involves assuming an object whose existence has formal similarities to that of a (parametrized) natural number object.
- To gain a complete understanding of solutions to differential equations, we will need to move to the more general setting of tangent *restriction* categories.

Tangent category definition

Definition (Rosický 1984, modified Cockett/Cruttwell 2013)

A **tangent category** consists of a category $\mathbb{X}$ with:

- **tangent bundle functor**: an endofunctor $T : \mathbb{X} \longrightarrow \mathbb{X}$;
- **projection of tangent vectors**: a natural transformation $p : T \longrightarrow 1_{\mathbb{X}}$;
- for each M, the pullback of n copies of p_M along itself exists (and is preserved by each T^m), call this pullback $T_n M$;
- **addition and zero tangent vectors**: for each $M \in \mathbb{X}$, p_M has the structure of a commutative monoid in the slice category $\mathbb{X}/M$; in particular there are natural transformations $+ : T_2 \longrightarrow T$, $0 : 1_{\mathbb{X}} \longrightarrow T$;

Tangent category definition (continued)

Definition

- **symmetry of mixed partial derivatives**: a natural transformation $c : T^2 \longrightarrow T^2$;
- **linearity of the derivative**: a natural transformation $\ell : T \longrightarrow T^2$;
- **the vertical bundle of the tangent bundle is trivial**:

$$\begin{array}{ccc} T_2(M) & \xrightarrow{\langle \pi_0 \ell, \pi_1 0_{TM} \rangle T(+)} & T^2(M) \\ {\scriptstyle \pi_0 p_M = \pi_1 p_M} \downarrow & & \downarrow {\scriptstyle T(p_M)} \\ M & \xrightarrow[0_M]{} & T(M) \end{array}$$

  is a pullback;
- various coherence equations for ℓ and c.

$\mathbb{X}$ is a **Cartesian tangent category** if $\mathbb{X}$ has products and T preserves them.

Examples

(i) Finite dimensional smooth manifolds with the usual tangent bundle.

(ii) Convenient manifolds with the kinematic tangent bundle.

(iii) Any Cartesian differential category (includes all Fermat theories by a result of MacAdam, and Abelian functor calculus by a result of Bauer et. al.).

(iv) The microlinear objects in a model of synthetic differential geometry (SDG).

(v) Commutative ri(n)gs and its opposite, as well as various other categories in algebraic geometry.

(vi) The category of C^∞-rings.

(vii) With additional pullback assumptions, tangent categories are closed under slicing.

Note: Building on work of Leung, Garner has shown how tangent categories are a type of enriched category.

Vector fields

Solving a differential equation is about turning a vector field into an *integral curve*, or, more generally, a *flow*.

Definition

A **vector field** on an object M is a section of the tangent bundle of M; that is, a map $F : M \longrightarrow TM$ such that $Fp_M = 1_M$.

Dynamical systems

Definition

A (parametrized) **dynamical system** on an object M consists of a vector field $F : M \longrightarrow TM$ and an "initial condition", i.e., a map $g : X \longrightarrow M$.

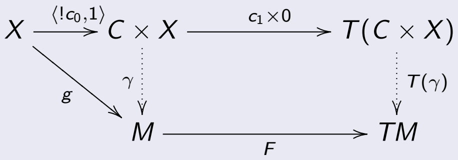
Total curve objects

Definition

A **total curve object** in a Cartesian tangent category consists of a dynamical system

$$1 \xrightarrow{c_0} C \xrightarrow{c_1} TC$$

which is initial in the following sense: for any other parametrized dynamical system $g : X \longrightarrow M, F : M \longrightarrow TM$, there is a unique map (the "solution") $\gamma : C \times X \longrightarrow M$ such that



Think of c_0 as "unit time" and c_1 as "unit speed".

Differential equations and curve object solutions

- For example, take $C = \mathbb{R}$ with $c_0 = 0$ and $c_1(x) = \langle 1, x \rangle$.
- Let F be a vector field on $M = \mathbb{R}$, so that $F(x) = \langle f(x), x \rangle$ for some smooth map $f : \mathbb{R} \longrightarrow \mathbb{R}$, and $z : \{\star\} \longrightarrow \mathbb{R}$ a point of $\mathbb{R}$.
- Then a solution γ as in the previous slide consists of a smooth map $\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\gamma(0) = z \text{ and } \gamma'(t) = f(\gamma(t)).$$

- In other words, to find such a γ one needs to solve the above (first-order, ordinary) differential equation.

Total curve objects: too restrictive

Example

In a model of SDG, D_∞ (the nilopotents of the ring object) is a total curve object (Kock/Reyes).

- But in a sense, these are "idealized" solutions: they only exist for an infinitesimal amount of time!
- For practical purposes, it is useful to understand how solutions work for some actual amount of time...
- $\mathbb{R}$ is *not* a total curve object in smooth manifolds:
  - solutions might "go off the edge";
  - solutions might "blow up".
- There is an existence and uniqueness theorem for differential equations, but solutions need only be partially defined!

Restriction categories

Restriction categories are a formalization of categories of partial maps due to Cockett and Lack:

Definition

A **restriction category** consists of a category $\mathbb{X}$, together with an operation which takes a map $f : A \longrightarrow B$ and produces a map $\overline{f} : A \longrightarrow A$ such that for $f : A \longrightarrow B$, $g : A \longrightarrow C$, $h : B \longrightarrow D$,

1. $\overline{f}\, f = f$;
2. $\overline{f}\, \overline{g} = \overline{g}\, \overline{f}$;
3. $\overline{\overline{g}\, f} = \overline{g}\, \overline{f}$;
4. $f\overline{h} = \overline{fh}\, f$.

- $\overline{f}$ is an idempotent which gives the "domain of definition of f".
- Say that f is **total** if $\overline{f} = 1$.

Tangent restriction categories

Definition

A **tangent restriction category** consists of a restriction category $\mathbb{X}$ with structure similar to that of a tangent category, and such that:

- $T : \mathbb{X} \longrightarrow \mathbb{X}$ preserves restrictions;
- all pullbacks are restriction pullbacks;
- the structural natural transformations $(p, +, 0, \ell, c)$ are all total.

Partial solutions

- We only expect that partial solutions need exist.
- In smooth manifolds, uniqueness can only be achieved on certain special types of “flow domains”.
- There are different ways of handling this axiomatically, but the way I’ll discuss here directly axiomatizes the existence of such domains.

Curve object definition

Definition

A **curve object** in a restriction tangent category consists of a total dynamical system

$$1 \xrightarrow{c_0} C \xrightarrow{c_1} TC$$

and, for each object X and restriction idempotent $e = \overline{e}$ on X, a collection of restriction idempotents called *definite domains*:

$$\mathcal{D}_e(X) \subseteq \{d = \overline{d} : C \times X \to C \times X, d \leq 1 \times e\}$$

such that:

- $\mathcal{D}_e(X)$ contains $1 \times e$ and is closed to intersections;
- for all $d \in \mathcal{D}_e(X)$, $\langle !c_0, e\rangle d = \langle !c_0, e\rangle$;
- for all $d \in \mathcal{D}_e(X)$ and $f : Y \to X$, $\overline{(1 \times f)d} \in \mathcal{D}_{\overline{f}}(Y)$;

Curve object definition continued

Definition

- (existence of solutions): every dynamical system (F, g) has a solution;
- (uniqueness of definite solutions): if γ and γ' are definite solutions to (F, g) then $\overline{\gamma} = \overline{\gamma'}$ implies $\gamma = \gamma'$;
- (density of definite solutions): for any solution α of a system (F, g) there is a definite solution γ of (F, g) such that $\gamma \leq \alpha$;
- (total linear solutions) if F is a linear vector field then any system (F, g) has a total solution.

If $\mathbb{X}$ has joins and each $\mathcal{D}_e(X)$ is closed under them, then each system has a unique **maximum** definite solution.

Curve object examples

Example

Any tangent category with a total curve object.

Example

$\mathbb{R}$ in the category of smooth manifolds.

Example

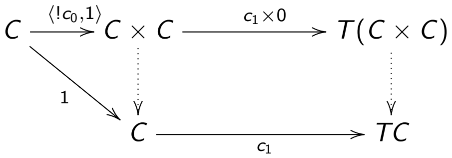
$\mathbb{R}$ in the category of Banach manifolds.

$\mathbb{R}$ is *not* a curve object in the category of convenient manifolds.

Curve object theory

With a curve object C, a number of standard results from differential geometry can be derived:

- If there is a total solution to $(c_1, 1)$:



  (call this solution $+$) then $(C, +, c_0)$ is a commutative monoid.
- If γ is a definite *flow* of a vector field $F : M \longrightarrow TM$ (i.e., a solution to $(F, 1)$ then there is a definite domain d on which

$$(+ \times 1)\gamma = (1 \times \gamma)\gamma.$$

- The flows of two vector fields "commute" if and only if they Lie bracket of their corresponding vector fields is 0.

Curve object theory continued

With a curve object C:

- **Higher-order ordinary differential equations** can be defined (they are certain vector fields on T^nM, ie., maps $T^{n-1}M \longrightarrow T^nM$) and their solutions exist.
- Connections have a corresponding notion of **parallel transport**: given a connection on a bundle $q : E \longrightarrow M$, any curve in M has a unique lift to a curve in E which stays "parallel" relative to the connection.
- Each connection on a tangent bundle has an associated notion of **geodesic**: given a tangent vector at a point, the particle traces out a path of "zero acceleration" (with "acceleration" relative to the connection).

Conclusions

In conclusion:

- The existence of solutions to differential equations can be formulated in tangent categories.
- The formulation is akin to adding a natural numbers object to a category.
- Many important results of differential geometry follow as a result of the assumption of such an object.
- The results allow one to simultaneously develop ideas for "infintesimal" solutions (as in SDG) and "actual" solutions (as in smooth finite-dimensional or Banach manifolds).

Future work

More work still to be done:

- More examples would be useful.
- Potential for further development of the theory (e.g. Frobenius' theorem).
- Development of other ways of handling uniqueness in the partial setting (unique "germinal" solutions).
- Partial differential equations is a whole other area that needs further exploration in this setting.

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