

# On flat 2-functors

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<sup>1</sup>Joint work with E. Dubuc and M. Szyld

# Flat functors and main theorem

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A commutative triangle diagram illustrating the relationship between a functor  $P$  and its left Kan extension  $P^*$ . The top vertex is  $A$ , the bottom vertex is  $\mathit{Set}$ , and the right vertex is  $\mathit{Hom}(A^{op}, \mathit{Set})$ . A solid arrow labeled  $P$  points from  $A$  to  $\mathit{Set}$ . A solid arrow labeled  $h$  points from  $A$  to  $\mathit{Hom}(A^{op}, \mathit{Set})$ . A dotted arrow labeled  $P^*$  points from  $\mathit{Set}$  to  $\mathit{Hom}(A^{op}, \mathit{Set})$ . A double-lined arrow labeled  $\eta_P$  points from  $P$  to  $P^*$ , indicating a natural isomorphism.

# Flat functors and main theorem

## Definition

$A \xrightarrow{P} \mathit{Set}$  is **flat** if its left Kan extension along the Yoneda embedding is left exact (preserves finite limits).

A commutative triangle diagram illustrating the left Kan extension of a flat functor  $P: A \rightarrow \mathit{Set}$ . The top horizontal arrow is  $h: A \rightarrow \mathit{Hom}(A^{op}, \mathit{Set})$ . The left vertical arrow is  $P: A \rightarrow \mathit{Set}$ . The right vertical arrow is  $P^*: \mathit{Hom}(A^{op}, \mathit{Set}) \rightarrow \mathit{Set}$ , shown as a dotted line. A horizontal arrow  $\eta_P: A \rightarrow \mathit{Hom}(A^{op}, \mathit{Set})$  is shown as a double-lined arrow, with a curved arrow indicating its relationship to  $h$ . A curved arrow also indicates the relationship between  $P$  and  $P^*$ .

## Theorem\*

For  $A \xrightarrow{P} \mathit{Set}$ , the following are equivalent:

- 1 The category of elements  $El_P$  of  $P$  is cofiltered.
- 2  $P$  is a filtered colimit of representable functors.
- 3  $P$  is flat.

\* [ML,M] Sheaves in Geometry and Logic: a First Introduction to Topos Theory, 1992.

# Idea of the proof

- ①  $El_P$  is cofiltered.
- ②  $P$  is a filtered colimit of representable functors.
- ③  $P$  is flat.

•  $1 \Rightarrow 2: P = \varinjlim_{El_P^{\circ P}} A(a, -) (= \int^a Pa \times A(a, -)).$

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Filtered colimit of flat is flat (since filtered colimits commute with finite limits).
- $3 \Rightarrow 1$ :  $C \xrightarrow{F} D$  left exact and  $C$  has finite limits  $\Rightarrow El_F$  cofiltered.  
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 $El_{P^*}$  cofiltered  $\Rightarrow El_P$  cofiltered.

# Flatness for $\mathcal{V}$ -enriched functors

A notion of flatness for  $\mathcal{V}$ -enriched functors was already considered by Kelly\*.

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There is no known generalization of the main theorem with this notion of flatness.

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Expressions for  $\mathcal{A} \xrightarrow{P} \mathcal{C}at$ 

There was no known expression of  $P$  as a (conical) colimit of representable functors.

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**Key Fact**

This pseudo-coend can be expressed as a special kind of (conical) colimit over the 2-category of elements  $\mathcal{E}l_P$  associated to  $P$ .

# Diamonds vs cones (dimension 1)

$$A \xrightarrow{P} \text{Set}, A^{op} \xrightarrow{F} \text{Set}.$$

Dicones for  $A \times A^{op} \xrightarrow{P \times F} \text{Set}$ :

$$\begin{array}{ccc}
 & Pa \times Fa & \\
 id \times Ff \nearrow & & \searrow \theta_a \\
 Pa \times Fb & \equiv & Z \\
 Pf \times id \searrow & & \nearrow \theta_b \\
 & Pb \times Fb & 
 \end{array}$$

Cones for  $El_P^{op} \xrightarrow{\diamond_P^{op}} A^{op} \xrightarrow{F} \text{Set}$ :

$$\begin{array}{ccc}
 Fa & & \\
 \lambda_{(x,a)} \searrow & & \nearrow \\
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$$\theta_a(x, y) = \lambda_{(x,a)}(y)$$

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$$\theta_a(x, y) = \lambda_{(x,a)}(y)$$

As a corollary,  $\int^a Pa \times Fa = \lim_{El_P^{op}} Fa.$

## Diamonds vs cones (dimension 2)

$$A \xrightarrow{P} \text{Set}, A^{op} \xrightarrow{F} \text{Set}.$$

Dicones for  $A \times A^{op} \xrightarrow{P \times F} \text{Set}$ :

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$$\mathcal{A} \xrightarrow{P} \mathit{Cat}, \mathcal{A}^{op} \xrightarrow{F} \mathit{Cat}.$$

Pseudodiamonds for  $\mathcal{A} \times \mathcal{A}^{op} \xrightarrow{P \times F} \mathit{Cat}$ :

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$\sigma$ -cones for  $\mathcal{E}l_P^{op} \xrightarrow{\diamond_P^{op}} \mathcal{A}^{op} \xrightarrow{F} \mathcal{C}at$ :

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 FA & & \\
 \lambda_{(x,A)} \searrow & & \\
 Ff \uparrow & \lambda_{(f,\varphi)} \nearrow & Z \\
 FB & \lambda_{(x',B)} \nearrow &
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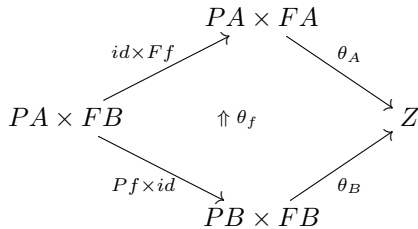
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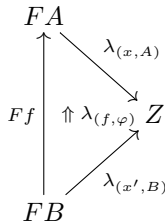
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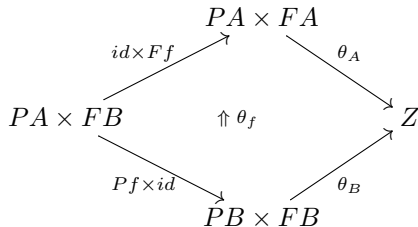
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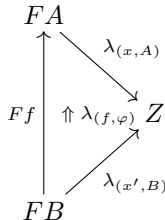
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As a corollary,  $p f^A PA \times FA = \underset{\mathcal{E}l_P^{op}}{\sigma\text{-lim}} FA.$

# Flat 2-functors and main theorem

## Definition

$A \xrightarrow{P} \mathbf{Set}$  is **flat** if its left Kan extension along the Yoneda embedding is left exact (preserves finite limits).

$$\begin{array}{ccc}
 A & \xrightarrow{h} & \mathbf{Hom}(A^{op}, \mathbf{Set}) \\
 & \searrow P & \nearrow P^* \\
 & & \mathbf{Set}
 \end{array}
 \quad
 \begin{array}{ccc}
 & \xrightarrow{\eta_P} & \\
 & & \swarrow \lrcorner \\
 & & \mathbf{Set}
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## Theorem

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- $1 \Rightarrow 2: P \approx \underset{\mathcal{E}l_P^{op}}{\sigma\text{-}lim} \mathcal{A}(A, -).$

- $2 \Rightarrow 3:$  Representable 2-functors are flat.

+

$\sigma$ -filtered  $\sigma$ -colimit of flat is flat (since  $\sigma$ -filtered  $\sigma$ -colimits commute with finite weighted bilimits\*).

- $3 \Rightarrow 1: C \xrightarrow{F} D$  left exact and  $C$  has finite limits  
 $\Rightarrow El_F$  cofiltered .

+

$El_{P^*}$  cofiltered  $\Rightarrow El_P$  cofiltered .

\*[DDS] A construction of certain weak colimits and an exactness property of the 2-category of categories, 2016.

## Idea of the proof

- ①  $\mathcal{E}l_P$  is  $\sigma$ -cofiltered.
- ②  $P$  is (equivalent to) a  $\sigma$ -filtered  $\sigma$ -colimit of representable 2-functors.
- ③  $P$  is flat.

- $1 \Rightarrow 2: P \approx \frac{\sigma\text{-}lim}{\mathcal{E}l_P^{op}} \mathcal{A}(A, -).$

- $2 \Rightarrow 3:$  Representable 2-functors are flat.

+

$\sigma$ -filtered  $\sigma$ -colimit of flat is flat (since  $\sigma$ -filtered  $\sigma$ -colimits commute with finite weighted bilimits\*).

- $3 \Rightarrow 1: \mathcal{C} \xrightarrow{F} \mathcal{D}$  left exact and  $\mathcal{C}$  has finite weighted bilimits  
 $\Rightarrow \mathcal{E}l_F$   $\sigma$ -cofiltered.

+

$$El_{P^*} \text{ cofiltered} \Rightarrow El_P \text{ cofiltered}.$$

\*[DDS] A construction of certain weak colimits and an exactness property of the 2-category of categories, 2016.

## Idea of the proof

- ①  $\mathcal{E}l_P$  is  $\sigma$ -cofiltered.
- ②  $P$  is (equivalent to) a  $\sigma$ -filtered  $\sigma$ -colimit of representable 2-functors.
- ③  $P$  is flat.

- $1 \Rightarrow 2$ :  $P \approx \frac{\sigma\text{-} \lim_{\mathcal{E}l_P^{op}} \mathcal{A}(A, -)}{\mathcal{E}l_P^{op}}$ .
- $2 \Rightarrow 3$ : Representable 2-functors are flat.  
+  
 $\sigma$ -filtered  $\sigma$ -colimit of flat is flat (since  $\sigma$ -filtered  $\sigma$ -colimits commute with finite weighted bilimits\*).
- $3 \Rightarrow 1$ :  $\mathcal{C} \xrightarrow{F} \mathcal{D}$  left exact and  $\mathcal{C}$  has finite weighted bilimits  
 $\Rightarrow \mathcal{E}l_F$   $\sigma$ -cofiltered.  
+  
 $\mathcal{E}l_{P^*}$   $\sigma$ -cofiltered  $\Rightarrow \mathcal{E}l_P$   $\sigma$ -cofiltered.

\*[DDS] A construction of certain weak colimits and an exactness property of the 2-category of categories, 2016.

Thank you!