On flat 2-functors

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[1] Joint work with E. Dubuc and M. Szyld

Flat functors and main theorem

Definition

$A \xrightarrow{P} Set$ is **flat** if its left Kan extension along the Yoneda embedding is left exact (preserves finite limits).

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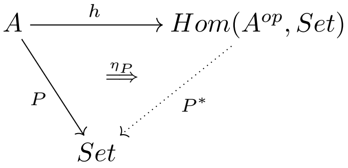

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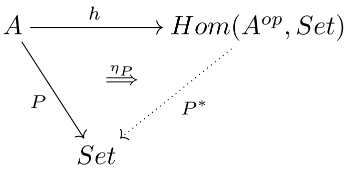

Theorem*

For $A \xrightarrow{P} Set$, the following are equivalent:

1. The category of elements El_P of P is cofiltered.
2. P is a filtered colimit of representable functors.
3. P is flat.

∗ [ML,M] Sheaves in Geometry and Logic: a First Introduction to Topos Theory, 1992.

Idea of the proof

1. El_P is cofiltered.
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- $1 \Rightarrow 2$: $P = \varinjlim_{El_P^{op}} A(a,-) \left(= \int^a Pa \times A(a,-)\right)$.

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  Filtered colimit of flat is flat (since filtered colimits commute with finite limits).
- $3 \Rightarrow 1$: $C \xrightarrow{F} D$ left exact and C has finite limits $\Rightarrow El_F$ cofiltered.

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  El_{P^*} cofiltered $\Rightarrow El_P$ cofiltered.

Flatness for $\mathcal{V}$-enriched functors

A notion of flatness for $\mathcal{V}$-enriched functors was already considered by Kelly*.

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$\mathcal{V} = \mathcal{C}at$ gives us a notion of flatness for 2-functors.

There is no known generalization of the main theorem with this notion of flatness.

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Expressions for $\mathcal{A} \xrightarrow{P} \mathcal{C}at$

There was no known expression of P as a (conical) colimit of representable functors.

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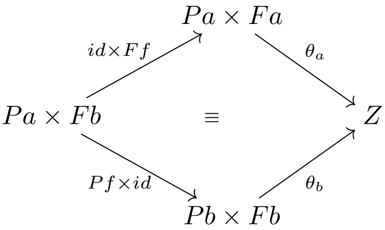
Key Fact

This pseudo-coend can be expressed as a special kind of (conical) colimit over the 2-category of elements $\mathcal{E}l_P$ associated to P.

Diamonds vs cones (dimension 1)

$A \xrightarrow{P} Set,\ A^{op} \xrightarrow{F} Set.$

Dicones for $A \times A^{op} \xrightarrow{P\times F} Set$:



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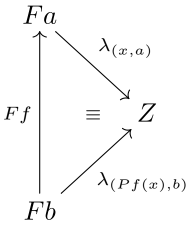

$$\theta_a(x,y) = \lambda_{(x,a)}(y)$$

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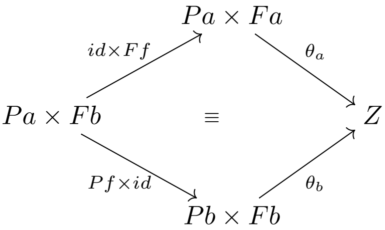

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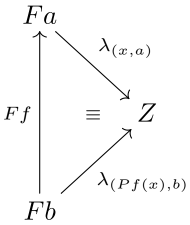

$$\theta_a(x, y) = \lambda_{(x,a)}(y)$$

As a corollary, $\int^a Pa \times Fa = \varinjlim_{El_P^{op}} Fa$.

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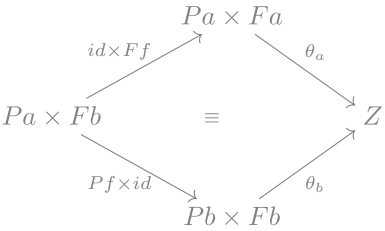

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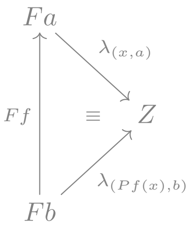

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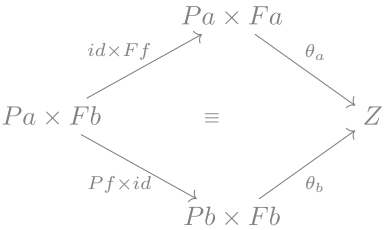

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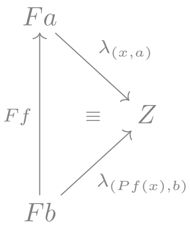

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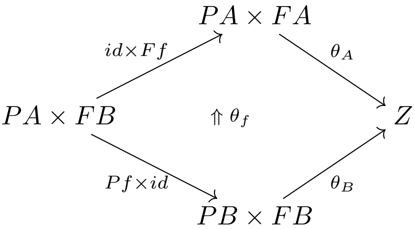

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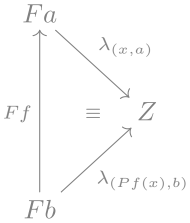

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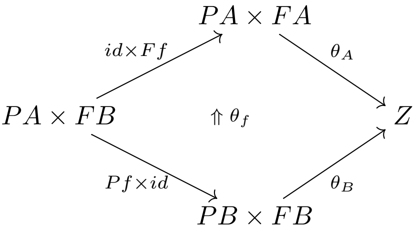

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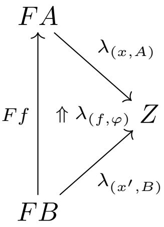

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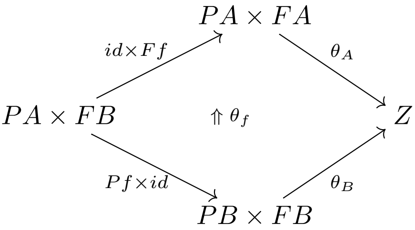

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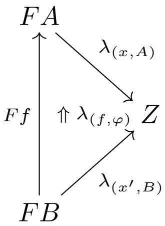

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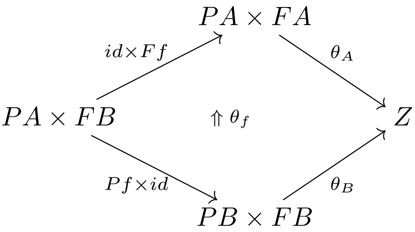

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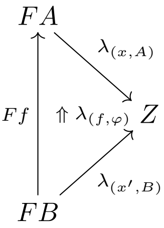

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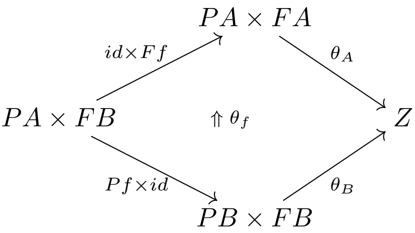

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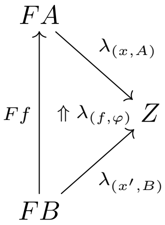

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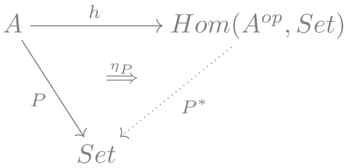

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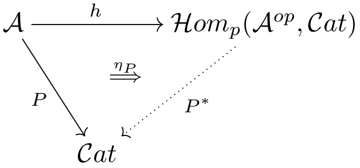

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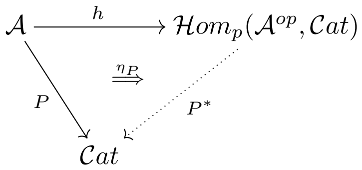

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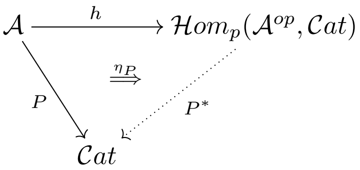

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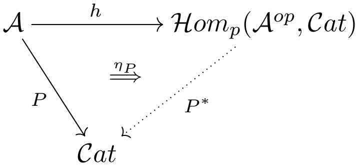

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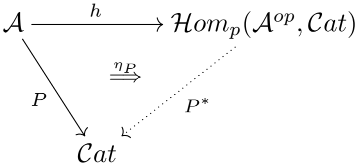

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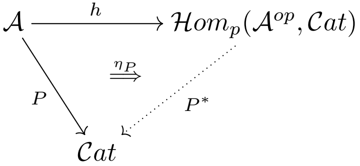

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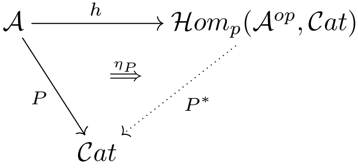

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1. $\mathcal{E}l_P$ is σ-cofiltered.
2. P is (equivalent to) a σ-filtered σ-colimit of representable 2-functors.
3. P is flat.

- $1 \Rightarrow 2$: $P \approx \underset{\mathcal{E}l_P^{op}}{\underrightarrow{\sigma\text{-}lim}}\, \mathcal{A}(A,-)$.
- $2 \Rightarrow 3$: Representable 2-functors are flat.

  $+$

  σ-filtered σ-colimit of flat is flat (since σ-filtered σ-colimits commute with finite weighted bilimits*).
- $3 \Rightarrow 1$: $\mathcal{C} \xrightarrow{F} \mathcal{D}$ left exact and $\mathcal{C}$ has finite weighted bilimits $\Rightarrow$ $\mathcal{E}l_F$ σ-cofiltered.

  $+$

  $\mathcal{E}l_{P^*}$ σ-cofiltered $\Rightarrow$ $\mathcal{E}l_P$ σ-cofiltered.

*[DDS] A construction of certain weak colimits and an exactness property of the 2-category of categories, 2016.

Thank you!