An Element-based Reformulation of Restriction Monads

Category Theory 2017 at University of British Columbia

Darien DeWolf
Dalhousie University

July 20, 2017

Restriction Categories

A restriction structure on a category **X** is an assignment of an arrow $\overline{f} : A \to A$ to each arrow $f : A \to B$ in **X** satisfying the following four conditions:

(R.1) For all maps f, $f\,\overline{f} = f$.

(R.2) For all maps $f : A \to B$ and $g : A \to B'$, $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$.

(R.3) For all maps $f : A \to B$ and $g : A \to B'$, $\overline{g\,\overline{f}} = \overline{g}\,\overline{f}$.

(R.4) For all maps $f : B \to A$ and $g : A \to B'$, $\overline{g}\,f = f\,\overline{gf}$.

A category equipped with a restriction structure is called a *restriction category*.

Restriction Monads: Definition Version 1

In a bicategory with involution, a restriction monad consists of a 0-cell x, 1-cells $T, D, E : x \to x$ and 2-cells

- $\eta : 1_T \Rightarrow T$,
- $\mu : T^2 \Rightarrow T$, $[\mu|_* DE] : DE \Rightarrow D$,
- $\rho : D \Rightarrow E$ (epic),
- $\iota : E \Rightarrow T$ (monic),
- $\Delta : T \Rightarrow TD$, $\tau : D^2 \Rightarrow D^2$ and
- $\psi : DT \Rightarrow TD$

satisfying conditions corresponding to (R.1) through (R.4) plus the usual monad laws plus $D^* D = DD^*$

Problem with the first approach

Ordinary monads in $\mathrm{Span}(\mathbf{Set})$ are in one-to-one correspondence with small categories.

Let $\mathbf{X}$ be a restriction category. We can easily construct a restriction monad $R(\mathbf{X})$ in $\mathrm{Span}(\mathbf{Set})$ with T, D, E behaving as desired, but we can't canonically go backwards: the D is not uniquely determined by the choice of T.

Problem with the first approach

Ordinary monads in $\mathrm{Span}(\mathbf{Set})$ are in one-to-one correspondence with small categories.

Let $\mathbf{X}$ be a restriction category. We can easily construct a restriction monad $R(\mathbf{X})$ in $\mathrm{Span}(\mathbf{Set})$ with T, D, E behaving as desired, but we can't canonically go backwards: the D is not uniquely determined by the choice of T.

One Solution:
Define restriction monads so that D and E naturally become "subobjects" of T by design. We can do this easily if we think of T as having elements and defining our operations on certain subsets of T.

Example: The elements of a monad in $\mathrm{Span}(\mathbf{Set})$.

Suppose that $\mathbf{X}$ is a small restriction category. For each element A of $\mathbf{X}_0$, we can define a span $\vec{A} : \{*\} \longrightarrow \mathbf{X}_0$ by

$$\{*\} \xleftarrow{\ \mathrm{id}\ } \{*\} \xrightarrow{\ A\ } \mathbf{X}_0$$

Peek-ahead: We will call such a span $\{*\}$-*elemental*.

Example: The elements of a monad in $\mathrm{Span}(\mathbf{Set})$.

Suppose that $\mathbf{X}$ is a small restriction category. Its corresponding monad in $\mathrm{Span}(\mathbf{Set})$ is of the form

$$\mathbf{X}_0 \xleftarrow{\;s\;} \mathbf{X}_1 \xrightarrow{\;t\;} \mathbf{X}_0$$

Example: The elements of a monad in $\mathrm{Span}(\mathbf{Set})$.

Suppose that $\mathbf{X}$ is a small restriction category. Its corresponding monad in $\mathrm{Span}(\mathbf{Set})$ is of the form

$$\mathbf{X}_0 \xleftarrow{\;s\;} \mathbf{X}_1 \xrightarrow{\;t\;} \mathbf{X}_0$$

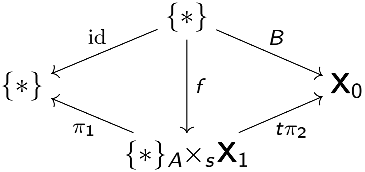
Composing $\vec{A}$ with T, then is of the form

$$\{*\} \xleftarrow{\;\mathrm{id}\;} \{*\}_A \times_s \mathbf{X}_1 \xrightarrow{\;t\pi_1\;} \mathbf{X}_0$$

Example: The elements of a monad in $\mathrm{Span}(\mathbf{Set})$.

$T\vec{A}$ contains as data all arrows of $\mathbf{X}$ with source A.

Given another object $B \in \mathbf{X}_0$, a span morphism $f : \vec{B} \longrightarrow T\vec{A}$, of the form



is therefore equivalent to the choice of an arrow f in $\mathbf{X}$ whose source is A and whose target is B.

$$\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \longleftrightarrow \mathbf{X}(A, B).$$

Example: The elements of a monad in $\mathrm{Span}(\mathbf{Set})$.

Such an identification allows us to define the restriction operator ρ as a family of set functions

$$\rho_{A,B} : \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \to \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A})$$

of arrows $f : A \to B$ to arrows $\rho(f) : A \to A$.

The conditions that this family of assignments satisfies will be given in a definition soon.

Example: The elements of a monad in $\mathrm{Span}(\mathbf{Set})$.

Identifying $\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A})$ with $\mathbf{X}(A, B)$, we must therefore consider how to "compose" elements of the set

$$\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \times \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{C}, T\vec{B}).$$

Example: The elements of a monad in $\mathrm{Span}(\mathbf{Set})$.

Identifying $\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A})$ with $\mathbf{X}(A, B)$, we must therefore consider how to "compose" elements of the set

$$\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \times \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{C}, T\vec{B}).$$

Note that such an element is of the form

$$\vec{C} \longrightarrow T\vec{B}$$
$$\vec{B} \longrightarrow T\vec{A},$$

For all $A, B, C \in \mathbf{X}_0$, define a composition map $\widetilde{\mu}$ to be a Kleisli-flavoured composite.

Example: The elements of a monad in $\mathrm{Span}(\mathbf{Set})$.

$\widetilde{\mu}$ is the composite:

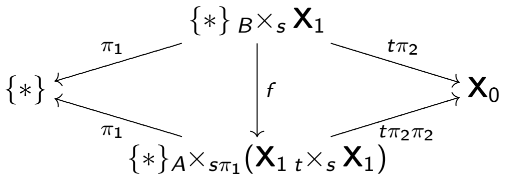
$$
\begin{array}{c}
\mathrm{Span}(\mathbf{Set})(\{*\},\mathbf{X}_0)(\vec{B},T\vec{A}) \times \mathrm{Span}(\mathbf{Set})(\{*\},\mathbf{X}_0)(\vec{C},T\vec{B}) \\
\Big\downarrow {\scriptstyle \mathrm{Span}(\mathbf{Set})(\{*\},\mathbf{X_0})(T,T)\times \mathrm{id}} \\
\mathrm{Span}(\mathbf{Set})(\{*\},\mathbf{X}_0)(T\vec{B},TT\vec{A}) \times \mathrm{Span}(\mathbf{Set})(\{*\},\mathbf{X}_0)(\vec{C},T\vec{B}) \\
\Big\downarrow {\scriptstyle \circ^{\mathrm{Span}(\mathbf{Set})(\{*\},\mathbf{X_0})}_{TT\vec{A},T\vec{B},\vec{C}}} \\
\mathrm{Span}(\mathbf{Set})(\{*\},\mathbf{X}_0)(\vec{C},TT\vec{A}) \\
\Big\downarrow {\scriptstyle \mathrm{Span}(\mathbf{Set})(\{*\},\mathbf{X_0})(\vec{C},\mu)} \\
\mathrm{Span}(\mathbf{Set})(\{*\},\mathbf{X}_0)(\vec{C},T\vec{A})
\end{array}
$$

Example: The elements of a monad in $\mathrm{Span}(\mathbf{Set})$.

Defining $\tilde{\mu}$ first requires an interpretation of the set

$$\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(T\vec{B}, TT\vec{A}).$$

Its elements are span morphisms of the form



These are assignments of arrows f with source B to composable pairs of arrows with source A and target tf :

$$(B \to C) \longmapsto (A \to C' \to C).$$

Example: The elements of a monad in $\mathrm{Span}(\mathbf{Set})$.

The morphism $\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(T, T)$

$$\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \longrightarrow \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(T\vec{B}, TT\vec{A})$$

is defined by

$$\left[f : A \to B\right] \longmapsto \left[(f, -) : (g : B \to C) \longmapsto (f : A \to B, g : B \to C)\right]$$

Example: The elements of a monad in $\mathrm{Span}(\mathbf{Set})$.

The morphism $\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(T, T)$

$$\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \longrightarrow \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(T\vec{B}, TT\vec{A})$$

is defined by

$$\left[f : A \to B\right] \longmapsto \left[(f, -) : (g : B \to C) \longmapsto (f : A \to B, g : B \to C)\right]$$

We can then compute this composite $\widetilde{\mu}$:

$$(f : A \to B, g : B \to C) \mapsto ((f, -), g) \mapsto (f, g) \mapsto \mu(f, g) = g \circ f;$$

The composition defined by μ coincides with $\widetilde{\mu}$ in $\mathrm{Span}(\mathbf{Set})$.

Restriction Monads: New Definition

Suppose that $\mathcal{B}$ is a bicategory with an involution on 1-cells containing an object E satisfying $\mathcal{B}(E, E)_0 \cong \mathcal{B}_0$.

Restriction Monads: New Definition

Suppose that $\mathcal{B}$ is a bicategory with an involution on 1-cells containing an object E satisfying $\mathcal{B}(E,E)_0 \cong \mathcal{B}_0$.

A restriction monad in $\mathcal{B}$ is a monad (T,η,μ) in $\mathcal{B}$ together with a family of functions

$$\rho_{A,B} : \mathcal{B}(E,x)(B,TA) \to \mathcal{B}(E,x)(A,TA)$$

indexed by E-elemental one-cells $A, B : E \to x$.
(**Definition:** E-elemental means: $A^* A \cong \mathrm{id}_E$.)

Restriction Monads: New Definition

Suppose that $\mathcal{B}$ is a bicategory with an involution on 1-cells containing an object E satisfying $\mathcal{B}(E, E)_0 \cong \mathcal{B}_0$.

A restriction monad in $\mathcal{B}$ is a monad (T, η, μ) in $\mathcal{B}$ together with a family of functions

$$\rho_{A,B} : \mathcal{B}(E, x)(B, TA) \to \mathcal{B}(E, x)(A, TA)$$

indexed by E-elemental one-cells $A, B : E \to x$.
(**Definition:** E-elemental means: $A^* A \cong \mathrm{id}_E$.)

Together with a way to compose morphisms between E-elemental one-cells, we impose four axioms on restriction monads, coming up.

Restriction Monads: New Definition

The way to compose morphisms between E-elemental one-cells is a multiplication map

$$\widetilde{\mu}_{A,B,C} : \mathcal{B}(E,x)(B,TA) \times \mathcal{B}(E,x)(C,TB) \to \mathcal{B}(E,x)(C,TA)$$

of E-elemental 1-cells defined by the composite

$$\left(\mathcal{B}(E,x)(T,T) \times \mathrm{id}\right); \left(\circ^{\mathcal{B}(E,x)}_{TTA,TB,C} \right); \mathcal{B}(E,x)(C,\mu_A).$$

For every triple of 1-cells $A, B, C : 1 \to x$, we require that the following diagrams commute which correspond to (R1)–(R4):

Restriction Monads: New Definition

(R.1) "$f\,\overline{f} = f$"

$$
\begin{array}{ccc}
\mathcal{B}(E,x)(B,TA) & \xrightarrow{\quad\Delta\quad} & \mathcal{B}(E,x)(B,TA) \times \mathcal{B}(E,x)(B,TA) \\
\Vert & & \downarrow{\scriptstyle \rho\times\mathrm{id}} \\
\mathcal{B}(E,x)(B,TA) & \xleftarrow[\quad\widetilde{\mu}\quad]{} & \mathcal{B}(E,x)(A,TA) \times \mathcal{B}(E,x)(B,TA)
\end{array}
$$

Restriction Monads: New Definition

(R.2) "$\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$"

$$\begin{array}{ccc}
\mathcal{B}(E,x)(B,TA) \times \mathcal{B}(E,x)(C,TA) & \xrightarrow{\rho\times\rho} & \mathcal{B}(E,x)(A,TA) \times \mathcal{B}(E,x)(A,TA) \\
\Big\downarrow{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle \widetilde{\mu}} \\
\mathcal{B}(E,x)(C,TA) \times \mathcal{B}(E,x)(B,TA) & & \mathcal{B}(E,x)(A,TA) \\
\Big\downarrow{\scriptstyle \rho\times\rho} & \nearrow{\scriptstyle \widetilde{\mu}} & \\
\mathcal{B}(E,x)(A,TA) \times \mathcal{B}(E,x)(A,TA) & &
\end{array}$$

Restriction Monads: New Definition

(R.3) "$\overline{g\,\overline{f}} = \overline{g}\,\overline{f}$"

$$\begin{array}{ccc}
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TA) & \xrightarrow{\rho\times\mathrm{id}} & \mathcal{B}(E,x)(A,TA)\times\mathcal{B}(E,x)(C,TA) \\
 & & \downarrow{\scriptstyle \widetilde{\mu}} \\
\downarrow{\scriptstyle \rho\times\rho} & & \mathcal{B}(E,x)(C,TA) \\
 & & \downarrow{\scriptstyle \rho} \\
\mathcal{B}(E,x)(A,TA)\times\mathcal{B}(E,x)(A,TA) & \xrightarrow[\widetilde{\mu}]{} & \mathcal{B}(E,x)(A,TA)
\end{array}$$

Restriction Monads: New Definition

(R.4) "$\overline{g}\, f = f\, \overline{gf}$"

$$
\begin{array}{ccccc}
\mathcal{B}(E,x)(B,TA) \times \mathcal{B}(E,x)(C,TB) & & \xrightarrow{\ \mathrm{id}\times\rho\ } & & \mathcal{B}(E,x)(B,TA) \times \mathcal{B}(E,x)(B,TB) \\
\downarrow{\scriptstyle \Delta\times\mathrm{id}} & & & & \\
\mathcal{B}(E,x)(B,TA) \times \mathcal{B}(E,x)(B,TA) \times \mathcal{B}(E,x)(C,TB) & & & & \downarrow{\scriptstyle \widetilde{\mu}} \\
\downarrow{\scriptstyle \mathrm{id}\times\widetilde{\mu}} & & & & \\
\mathcal{B}(E,x)(B,TA) \times \mathcal{B}(E,x)(C,TA) & \xrightarrow[\mathrm{id}\times\rho]{} & \mathcal{B}(E,x)(B,TA) \times \mathcal{B}(E,x)(A,TA) & \xrightarrow[\widetilde{\mu}.\tau]{} & \mathcal{B}(E,x)(B,TA)
\end{array}
$$

Restriction Monads as Internal Categories

Proposition

Small restriction categories are in one-to-one correspondence with restriction monads in $\mathrm{Span}(\mathbf{Set})$.

Restriction Monads as Internal Categories

Proposition

Small restriction categories are in one-to-one correspondence with restriction monads in $\mathrm{Span}(\mathbf{Set})$.

Proposition

Let $\mathbf{C}$ *be a category with all pullbacks over* s *and* t*. Restriction monads in* $\mathrm{Span}(\mathbf{C})$ *are in one-to-one correspondence with restriction categories internal to* $\mathbf{C}$.

Restriction Monads as Enriched Categories

Proposition

Restriction monads in **Set**-Mat *are in one-to-one correspondence with small restriction categories.*

Restriction Monads as Enriched Categories

Proposition

Restriction monads in **Set**-Mat *are in one-to-one correspondence with small restriction categories.*

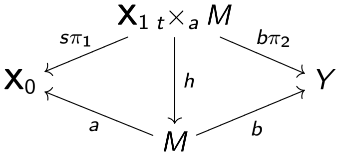
Proposition

If $\mathcal{V}$ is a Cartesian monoidal category, then restriction $\mathcal{V}$-categories are restriction monads in $\mathcal{V}$-Mat.

Algebras for Restriction Monads

Let T be an ordinary monad in $\mathrm{Span}(\mathbf{Set})$ and that $\mathbf{X}$ is its corresponding small category.

Recall that algebras (S, h) for T are right-$\mathbf{X}$ modules on the apex set of $S = \mathbf{X}_0 \xleftarrow{a} M \xrightarrow{b} Y$ with the action given by $h : ST \Rightarrow S$:



Similarly to how we identify the hom-set $\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A})$ with $\mathbf{X}(A, B)$, we identify $\mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$ with the module set $S(B, A)$.

Similarly to how we identify the hom-set $\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A})$ with $\mathbf{X}(A, B)$, we identify $\mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$ with the module set $S(B, A)$.

We can then define a restriction operator r as a family of set functions

$$r_{A,B} : \mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A}) \to \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A})$$

$$\alpha : A \longrightarrow B \longmapsto r(\alpha) : A \to A$$

Similarly to how we identify the hom-set $\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A})$ with $\mathbf{X}(A, B)$, we identify $\mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$ with the module set $S(B, A)$.

We can then define a restriction operator r as a family of set functions

$$r_{A,B} : \mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A}) \to \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A})$$

$$\alpha : A \longrightarrow B \longmapsto r(\alpha) : A \to A$$

We will require that each $r(\alpha)$ is a restriction idempotent of $\mathbf{X}$:

$$\mathrm{Im}(r_{A,B}) \subseteq \cup_{A':1 \to X} \mathrm{Im}(\rho_{A,A'})$$

Identifying $\mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$ with $S(B, A)$, we must therefore consider how to "h-act" with elements of the set

$$\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A}') \times \mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$$

Much like $\widetilde{\mu}$, we will define $\widetilde{h}$ as a Kleisli-styled composite which will coincide (on the hom-categories) with h in $\mathrm{Span}(\mathbf{Set})$.

There will also be restriction axiom diagrams which will have to commute.

$\widetilde{h}$ is the composite:

$$\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A}') \times \mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$$

$$\Big\downarrow \ \mathrm{Span}(\mathbf{Set})(\{*\}, S\mathbf{X_0})(S,S) \times \mathrm{id}$$

$$\mathrm{Span}(\mathbf{Set})(\{*\}, Y)(S\vec{A}, ST\vec{A}') \times \mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$$

$$\Big\downarrow \ \circ^{\mathrm{Span}(\mathbf{Set})(\{*\},Y)}_{ST\vec{A}', S\vec{A}, \vec{B}}$$

$$\mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, ST\vec{A}')$$

$$\Big\downarrow \ \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X_0})(\vec{B}, h)$$

$$\mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A}')$$

(R.1)

$$\begin{array}{ccc}
\mathcal{B}(E,y)(B,SA) & \xrightarrow{\Delta} & \mathcal{B}(E,y)(B,SA) \times \mathcal{B}(E,y)(B,SA) \\
\| & & \downarrow r\times\mathrm{id} \\
\mathcal{B}(E,y)(B,SA) & \xleftarrow[\widetilde{h}]{} & \mathcal{B}(E,x)(A,TA) \times \mathcal{B}(E,y)(B,SA)
\end{array}$$

(R.3)

$$\begin{array}{ccc}
\mathcal{B}(E,y)(B,SA)\times\mathcal{B}(E,y)(B',SA) & \xrightarrow{r\times\mathrm{id}} & \mathcal{B}(E,x)(A,TA)\times\mathcal{B}(E,y)(B',SA) \\
 & & \downarrow{\scriptstyle \widetilde{h}} \\
\downarrow{\scriptstyle r\times r} & & \mathcal{B}(E,y)(B',SA) \\
 & & \downarrow{\scriptstyle r} \\
\mathcal{B}(E,x)(A,TA)\times\mathcal{B}(E,x)(A,TA) & \xrightarrow[\widetilde{\mu}]{} & \mathcal{B}(E,x)(A,TA)
\end{array}$$

(R.4)

$$\begin{array}{ccc}
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TB) & \xrightarrow{\mathrm{id}\times r} & \mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(B,TB) \\
\downarrow{\scriptstyle \Delta\times\mathrm{id}} & & \\
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TB) & & \downarrow{\scriptstyle \widetilde{\mu}} \\
\downarrow{\scriptstyle \mathrm{id}\times\widetilde{h}} & & \\
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TA) & & \\
\downarrow{\scriptstyle \mathrm{id}\times r} & & \\
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(A,TA) & \xrightarrow[\widetilde{\mu}.\tau]{} & \mathcal{B}(E,x)(B,TA)
\end{array}$$

Proposition

Let $\mathbf{X}$ *be a small restriction category and let* (T, η, μ, ρ) *denote its corresponding restriction monad in* $\mathrm{Span}(\mathbf{Set})$. *An algebra*

$$(S = \mathbf{X}_0 \xleftarrow{a} M \xrightarrow{b} Y, h, r)$$

is a right-X *restriction module, whose right* $\mathbf{X}$*-action is defined by* h*-evaluation.*