FROBENIUS AND HOPF $\mathcal{V}$-CATEGORIES

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Overview

- Preliminaries
- Frobenius and Hopf $\mathcal{V}$-categories
- Special case: $\mathcal{V} = \mathsf{Mod}_k$
- Equivalent definitions of Frobenius $\mathcal{V}$-categories
- Larson Sweedler theorem

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- $(\mathcal{V}, \otimes, I) = (\text{Sets}, \times, \{*\})$: ordinary categories
- $(\mathcal{V}, \otimes, I) = (\text{Mod}_k, \otimes_k, k)$: k-linear categories

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Definition

If $(\mathcal{V}, \otimes, I, \sigma)$ is braided, a *semi-Hopf $\mathcal{V}$-category H* is a Comon($\mathcal{V}$)-enriched category. Explicitly, it consists of a collection of objects X and for every $x, y \in X$ an object $H_{x,y}$ of $\mathcal{V}$, together with families of morphisms in $\mathcal{V}$

$$m_{xyz} : H_{x,y} \otimes H_{y,z} \to H_{x,z} \qquad j_x : I \to H_{x,x}$$

which make H a $\mathcal{V}$-category, and a collection of morphisms

$$c_{xy} : H_{x,y} \to H_{x,y} \otimes H_{x,y} \qquad \epsilon_{xy} : H_{x,y} \to I$$

such that each $(H_{x,y}, c_{xy}, \epsilon_{xy})$ is a comonoid in $\mathcal{V}$ and satisfying the following equations:

$$
\begin{array}{ccc}
H_{x,y} \otimes H_{y,z} & \xrightarrow{c_{xy} \otimes c_{yz}} & H_{x,y} \otimes H_{x,y} \otimes H_{y,z} \otimes H_{y,z} \\
 & & \downarrow{\scriptstyle H_{x,y} \otimes \sigma \otimes H_{y,z}} \\
\downarrow{\scriptstyle m_{xyz}} & & H_{x,y} \otimes H_{y,z} \otimes H_{x,y} \otimes H_{y,z} \\
 & & \downarrow{\scriptstyle m_{xyz} \otimes m_{xyz}} \\
H_{x,z} & \xrightarrow[c_{xz}]{} & H_{x,z} \otimes H_{x,z}
\end{array}
$$

$$
\begin{array}{ccc}
I & \xrightarrow{\sim} & I \otimes I \\
\downarrow{\scriptstyle j_x} & & \downarrow{\scriptstyle j_x \otimes j_x} \\
H_{x,x} & \xrightarrow[c_{xx}]{} & H_{x,x} \otimes H_{x,x}
\end{array}
\qquad
\begin{array}{ccc}
H_{x,y} \otimes H_{y,z} & \xrightarrow{\epsilon_{xy} \otimes \epsilon_{yz}} & I \otimes I \\
\downarrow{\scriptstyle m_{xyz}} & & \downarrow{\scriptstyle \sim} \\
H_{x,z} & \xrightarrow[\epsilon_{xz}]{} & I
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{\mathrm{id}} & I \\
\downarrow{\scriptstyle j_x} & & \downarrow{\scriptstyle \mathrm{id}} \\
H_{x,x} & \xrightarrow[\epsilon_{xx}]{} & I
\end{array}
$$

Definition

A *Hopf $\mathcal{V}$-category* is a semi-Hopf $\mathcal{V}$-category equipped with a family of maps $S_{xy} : H_{x,y} \to H_{y,x}$ satisfying

$$
\begin{array}{ccccc}
 & H_{x,y} \otimes H_{x,y} & \xrightarrow{H_{x,y}\otimes S_{xy}} & H_{x,y} \otimes H_{y,x} & \\
{}^{c_{xy}}\nearrow & & & & \searrow^{m_{xyx}} \\
H_{x,y} & \xrightarrow{\qquad \epsilon_{xy} \qquad} & I & \xrightarrow{\qquad j_x \qquad} & H_{x,x}
\end{array}
$$

$$
\begin{array}{ccccc}
 & H_{x,y} \otimes H_{x,y} & \xrightarrow{S_{xy}\otimes H_{x,y}} & H_{y,x} \otimes H_{x,y} & \\
{}^{c_{xy}}\nearrow & & & & \searrow^{m_{yxy}} \\
H_{x,y} & \xrightarrow{\qquad \epsilon_{xy} \qquad} & I & \xrightarrow{\qquad j_y \qquad} & H_{y,y}\ .
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This family of maps is called the *antipode* of H.

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Every Hopf algebra H in a braided monoidal $\mathcal{V}$ is a 1-object Hopf $\mathcal{V}$-category;

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Example

Let A be a k-linear Hopf category, with $\mathrm{Ob(A)} = \mathrm{X}$ a finite set. Then $\hat{A} = \bigoplus_{x,y \in X} A_{x,y}$ is a weak Hopf algebra.

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Definition

A *Frobenius $\mathcal{V}$-category A* is a $\mathcal{V}$-category that is also a $\mathcal{V}$-opcategory and satisfies indexed 'Frobenius conditions'. Explicitly, it consists of a set of objects X and for every $x, y \in X$ an object $A_{x,y}$ of $\mathcal{V}$ together with maps

$$m_{xyz} : A_{x,y} \otimes A_{y,z} \to A_{x,z} \qquad j_x : I \to A_{x,x}$$
$$d_{abc} : A_{a,c} \to A_{a,b} \otimes A_{b,c} \qquad e_y : A_{y,y} \to I$$

that make A a $\mathcal{V}$-category and a $\mathcal{V}$-opcategory, and satisfy the following axioms.

$$
\begin{array}{ccccc}
A_{x,y} \otimes A_{y,z} & & \xrightarrow{\;d_{xwy} \otimes 1\;} & & A_{x,w} \otimes A_{w,y} \otimes A_{y,z} \\
\Big\downarrow {\scriptstyle 1 \otimes d_{ywz}} & \searrow^{m_{xyz}} & & & \Big\downarrow {\scriptstyle 1 \otimes m_{wyz}} \\
 & & A_{x,z} & \searrow^{d_{xwz}} & \\
A_{x,y} \otimes A_{y,w} \otimes A_{w,z} & & \xrightarrow[\;m_{xyw} \otimes 1\;]{} & & A_{x,w} \otimes A_{w,z}
\end{array}
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Every Frobenius monoid in a monoidal category $\mathcal{V}$ is a one-object Frobenius $\mathcal{V}$-category.

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For a field k, let Mat be the category whose objects are the natural numbers and whose hom-sets $\mathrm{Mat}_{m,n}$ are the sets of $m \times n$ matrices with entries in k. This is a Mod_k-category if we take the usual composition of matrices and the usual identity matrices. The comultiplication and counit are defined by

$$d_{n,p,m} \colon \mathrm{Mat}_{n,m} \to \mathrm{Mat}_{n,p} \otimes \mathrm{Mat}_{p,m} \colon e_{i,j} \mapsto \sum_{t=1}^{p} e_{i,t} \otimes e_{t,j}$$

$$\epsilon_{n,m} \colon \mathrm{Mat}_{n,m} \to k \colon e_{i,j} \mapsto \delta_{i,j}$$

for all $1 \leq i \leq n, 1 \leq j \leq m$, with $e_{*,*}$ the usual basis of $\mathrm{Mat}_{*,*}$. This makes Mat into a Frobenius category.

Proposition

Let A be a Frobenius $\mathcal{V}$-category with a finite set X of objects, where $\mathcal{V}$ is a semi-additive category (i.e. a category for which finite products and coproducts coincide) Then the 'packed form' $\hat{A} = \coprod_{x,y \in X} A_{x,y}$ of A is a Frobenius monoid in $\mathcal{V}$.

Equivalent definitions of Frobenius $\mathcal{V}$-category

From now on $\mathcal{V} = \mathrm{Mod}_k$

Definition

Let (A, m, j) be a k-linear category with $\mathrm{Ob}A = X$. A *Casimir family* E is a X^2-indexed family of elements $\overset{xy}{e} = e^1_{x,y} \otimes e^2_{y,x} \in A_{x,y} \otimes A_{y,x}$ such that for every $a \in A_{x,z}$

$$a e^1_{z,y} \otimes e^2_{y,z} = e^1_{x,y} \otimes e^2_{y,x} a \ .$$

Proposition

A $\mathcal{V}$-category A is Frobenius if and only if there exists a Casimir family E and morphisms $\nu_x : A_{x,x} \to k$ for all $x \in X$, such that for each $\overset{xx}{e} \in E$:

$$\nu_x(e^1_{x,x}) \cdot e^2_{x,x} = e^1_{x,x} \cdot \nu_x(e^2_{x,x}) = 1_{x,x}$$

for all $x \in X$.

We call (E, ν) a Frobenius system of the $\mathcal{V}$-category A.

Proposition

For a k-linear category A, the following assertions are equivalent.

1. *A is Frobenius k-linear category.*
2. - *$A_{x,y}$ is finitely generated projective as k-module for every $x, y \in X$; and*
   - *A and $(A^*)^{op}$ are isomorphic as right (or left) A-modules.*

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Remark

- The right module structure is given by: $A^*_{x,y} \otimes A_{y,z} \to A^*_{x,z} : g \otimes a \mapsto g(a-)$
- We are working in Mod_k^f!

Definition

The *left integral* $\int_{A_{x,y}}^{l}$ for the hom-object $A_{x,y}$ is the collection

$$\int_{A_{x,y}}^{l} = \left\{ t \in A_{x,y} \mid ht = \epsilon_{x,x}(h) \cdot t, \forall h \in A_{x,x} \right\},$$

We denote the family of all (x, y)-left integrals for A by $\int_{A}^{l}$.

Larson-Sweedler theorem

Theorem

Let $(H, m, j, c, \epsilon, S)$ be a k-linear Hopf category. The following assertions are equivalent:

1. *(H, m, j) has the structure of a Frobenius k-linear category;*
2. *H is locally finitely generated and projective, and all k-modules $\int^{l}_{H^*_{x,x}}$ are free of rank one.*

Example

Let H be a k-linear Hopf category with a finite set of objects X.
Recall:

- $\bigoplus_{x,y \in X} H_{x,y}$ is a weak Hopf algebra
- The packed form of a Frobenius k-linear category is a Frobenius monoid in Mod_k

The previous theorem now gives us a class of weak Hopf algebras that are also Frobenius.

Thank you!

References

E. Batista, S. Caenepeel, and J. Vercruysse. Hopf Categories. Algebr. Represent. Theory, 19(5):1173-1216, 2016.