Towards a realizability model of homotopy type theory

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Overview

- **Motivation** : construct *realizability model* of *homotopy type theory*, to show consistency of *impredicative univalent universe*
- **Approach** : internalize *cubical set model* in Hyland's effective topos $\mathcal{E}\mathit{ff}$
- **Context** : build on related work by Coquand et al., Orton/Pitts, Gambino/Sattler, Frumin/van den Berg, Rosolini

Homotopy type theory :

Re-reading of Martin-Löf's **dependent type theory** where

1. **types** are **spaces**
2. **equalities** are **paths**

... more precisely :

Dependent Type Theory

Dependent type theory comprises:

- simple types $1,\ X,\ Y,\ A \times B,\ A \Rightarrow B,\ A + B,\ \ldots$
- dependent types / type families $x : A \vdash B(x)$
- dependent sum types $\Sigma x : A \,.\, B(x)$ and product types $\Pi x : A \,.\, B(x)$
- inductive types $\mathbb{N},\ \mathsf{list}(A),\ \ldots$
- identity types $x : A, y : A \vdash \mathrm{Id}_A(x, y)$
- universes $\mathcal{U}$ which are 'types of types', closed under the preceding type constructors

Identity types

In the set-theoretic model, identity types are given by

$$\mathsf{Id}_A(x, y) = \begin{cases} \{*\} & \text{if } x = y \\ \varnothing & \text{else} \end{cases} .$$

In **locally cartesian closed categories**, identity types are modeled by diagonals

$$A \to A \times A.$$

Interpretation satisfies **uniqueness of identity proofs**

$$\text{(UIP)} \qquad \Pi a, b : A \,.\, \Pi p, q : \mathsf{Id}_A(a, b) \,.\, \mathrm{Id}_{\mathsf{Id}_A(a,b)}(p, q),$$

not provable in type theory (Hofmann-Streicher 1994).

Irritating to classical mathematicians, but leaves room for a **homotopical interpretation**.

Identity types as path objects

Awodey-Warren (2009): interpret Id-types by **fibrant replacement** of diagonal, i.e. second part of a trivial-cofibration/fibration factorization

$$\begin{tikzcd} A \arrow[r, hook, "\sim"] \arrow[dr] & PA \arrow[d, two heads, "\mathrm{Id}_A"] \\ & A \times A \end{tikzcd}$$

w.r.t. a **weak factorization system / WFS** (possibly part of a model structure).

Intuition: elements of $\mathsf{Id}_A(a, b)$ are **paths** from a to b.

Lifting property of WFS corresponds to **elimination rule** of Id-types.

Coherence problem solved by using ‘categories-with-families’ and cloven WFS.

h-levels and equivalences

Types satisfying UIP can be recovered as 0**-types** in HoTT.

More generally, n**-types** for $n \geq -2$ are inductively defined as follows:

- A is a (-2)**-type** (or **contractible type**), if

  $$\Sigma x : A \, . \, \Pi y : A \, . \, \mathsf{Id}_A(x, y)$$

  is inhabited.
- A is a $(n+1)$**-type**, if $\mathsf{Id}_A(x, y)$ is an n-type for all $x, y : A$.

We call (-1)-types **propositions**, and 0-types **sets**.

Equivalences

A function $f : A \to B$ is called an **equivalence**, if its **fibers**

$$\Sigma x : A \,.\, \mathrm{Id}_B(fx, y)$$

are contractible for all $y : B$.
$\mathrm{equiv}(A, B)$ is the type of equivalences from A to B.

Universes and univalence

When should two types be considered equal?

Voevodsky's **univalence axiom** asserts that two types are equal iff they are homotopy equivalent.

More precisely, a universe $\mathcal{U}$ is called **univalent**, if the canonical map

$$\mathsf{Id}_{\mathcal{U}}(A, B) \to \mathsf{equiv}(A, B)$$

is an equivalence for all $A, B : \mathcal{U}$.

Univalence is inconsistent with UIP as soon as a type in $\mathcal{U}$ has a non-trivial automorphism.

Since classical logic implies “proof-irrelevance”, it is inconsistent with univalence.

A model of HoTT with univalent universe in simplicial sets has been descibed by Voevodsky, written down by Kapulkin-Lumsdaine 2012.

Predicative and impredicative universes

- Ordinary **predicative** universes are closed under **small** products of small types:

$$\frac{\Gamma \vdash A : \mathcal{U} \qquad \Gamma, x : A \vdash B(x) : \mathcal{U}}{\Gamma \vdash \Pi x : A \,.\, B(x) : \mathcal{U}}$$

- **Impredicative** universes are closed under **arbitrary** products of small types:

$$\frac{\Gamma, x : A \vdash B(x) : \mathcal{U}}{\Gamma \vdash \Pi x : A \,.\, B(x) : \mathcal{U}}$$

- Subobject classifier Ω of a topos models impredicative universe of propositions.
- Impredicative universe $\mathcal{U}$ containing a type $A : \mathcal{U}$ with two distinct elements $x \neq y : A$ inconsistent with classical logic.

Impredicative universes in realizability toposes

The **effective topos** $\mathcal{E}\!f\!f$ (Hyland 1980) models an impredicative universe $\mathcal{M}$ containing non-propositional types.

$\mathcal{M}$ is not univalent (since in topos-models, all types are 0-types)

To get an **univalent, impredicative universe**, need something like

- **homotopical realizability model** or
- **realizability-∞-topos**

Constructing the model internally to $\mathcal{E}ff$

Observation : Existence of univalent universe in simplicial set model relies on assumption of Grothendieck universe in meta-theory.

Idea : perform model construction internally to $\mathcal{E}ff$ (containing impredicative universe) to obtain **univalent** impredicative universe.

Working internally to $\mathcal{E}ff$ imposes restrictions:

- constructive internal logic (no excluded middle)
- no transfinite constructions (no ‘small object argument’)

Coquand et al observed that the simplicial model relies on classical logic, proposed to use **cubical sets** instead.

Cubical sets

Cubical sets are presheaves on a **cube category**

- **Monoidal cube category** $\mathbb{C}_{\mathsf{m}}$ used by Serre, Kan in 50ies
- **Symmetric cube category** $\mathbb{C}_{\mathsf{s}}$: free symmetric monoidal category on an interval (Bezem, Coquand, Huber 2013)
- **Cartesian cube category** $\mathbb{C}_{\mathsf{c}}$: free finite-product category on an interval / Lawvere theory with two constants
- **Cartesian cube category with connections** $\mathbb{C}_{\mathsf{cc}}$: Lawvere theory of distributive lattices / full subcat of **Cat** on objects $\mathbb{2}^n$
- **Lawvere theory of de Morgan algebras** $\mathbb{C}_{\mathsf{dm}}$ (Cohen, Coquand, Huber, Mörtberg 2016)

Comparison :

- all locally finite & can be internalized in $\mathcal{E}\mathit{ff}$
- use $\mathbb{C}_{\mathsf{c}}$ or $\mathbb{C}_{\mathsf{cc}}$
- $\mathbb{C}_{\mathsf{c}}$ much simpler than $\mathbb{C}_{\mathsf{cc}}$:

$$\#\mathbb{C}_{\mathsf{c}}([9],[1]) = 11 \qquad \#\mathbb{C}_{\mathsf{cc}}([9],[1]) = ?$$ (9th Dedekind number)

More on cube categories :
"Varieties of cubical sets" – Buchholtz, Morehouse 2017

(Iterated) path spaces in cartesian cubical sets $\widehat{\mathbb{C}}$

$[0], [1], [2], \ldots$ objects of cube category.

Interval : $I = Y([1])$

n-cube : $I^n = Y([n]) = Y([1])^n$

Path object : $PA = A^I = A(- \times [1])$

Iterated path object : $P^n A = A^{I^n} = A(- \times [n])$

(I **tiny object**, $A \mapsto A^I$ has right adjoint – ‘fractional exponent’)

Path space factorization

Awodey 2016 : algebraic weak factorization system (AWFS) on $\widehat{\mathbb{C}_c}$ such that

$$\begin{tikzcd} A \arrow[r, "\tilde{\pi}"] \arrow[dr] & A^I \arrow[d, "\langle A^\perp, A^\top \rangle"] \\ & A \times A \end{tikzcd}$$

is an (L, R)-factorization.

Construction uses small objects argument

To avoid this and be able to internalize in $\mathcal{E}ff$, restrict to **Kan complexes**.

Uniform normal Kan complexes

Box inclusions analogous to simplicial Horn inclusions :

$$\sqcup_j^n \hookrightarrow I^n \qquad\qquad n \in \mathbb{N},\ j \in \{\bot, \top\}$$

Uniform Kan complexes have coherently chosen box fillings :

$$\begin{tikzcd}
X \times \sqcup_j^n \arrow[r, "f"] \arrow[d, hook] & A \\
X \times I^n \arrow[ur, dashed, "\tilde{f}"'] &
\end{tikzcd}$$

Normality condition : fillers of 'degenerate boxes' are degenerate

$$\begin{tikzcd}
\sqcup_j^n \arrow[rr] \arrow[dr] \arrow[dd, hook] & & A \\
& I^{n-1} \arrow[ur] & \\
I^n \arrow[ur] \arrow[uurr, dashed, bend right] & &
\end{tikzcd}$$

$\mathcal{F}(\widehat{\mathbb{C}_c}) \subseteq \widehat{\mathbb{C}_c}$ category of uniform normal Kan complexes

Cloven weak factorization systems

A **cloven weak factorization system / CWFS** (van den Berg, Garner 2010) on $\mathcal{C}$ is a functorial factorization

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
X \arrow[r, "k"'] & Y
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "Lf"'] & B \arrow[d, "Lg"] \\
P(f) \arrow[r, "{P(h,k)}"] \arrow[d, "Rf"'] & P(g) \arrow[d, "Rg"] \\
X \arrow[r, "k"'] & Y
\end{tikzcd}
$$

with specified fillers for all $f : A \to B$ (no naturality requirement):

$$
\begin{tikzcd}
A \arrow[r, "LLf"] \arrow[d, "Lf"'] & P(Lf) \arrow[d, "RLf"] \\
Pf \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed] & Pf
\end{tikzcd}
\qquad
\begin{tikzcd}
Pf \arrow[r, "\mathrm{id}"] \arrow[d, "LRf"'] & Pf \arrow[d, "Rf"] \\
P(Rf) \arrow[r, "RRf"'] \arrow[ur, dashed] & B
\end{tikzcd}
$$

Theorem

CWFS with stable functorial choice of diagonal factorization gives rise to model of Id-types.

A cloven CWFS on $\mathcal{F}(\widehat{\mathbb{C}_{\mathsf{c}}})$

The mapping-cocylinder factorization on uniform Kan complexes gives a cloven CWFS satisfying the conditions of the theorem :

$$
\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "Lf"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
 & Pf \arrow[r] \arrow[d] \arrow[dl, "Rf"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^I \arrow[d] \\
B & A \times B \arrow[l, "\pi_2"] \arrow[r, "f \times B"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \times B \arrow[d, "\pi_1"] \\
 & A \arrow[r, "f"] & B
\end{tikzcd}
$$

The induced WFS

Every CWFS induces a WFS with left maps L-coalgebras and right maps R-algebras.

Theorem

TFAE for $i : U \to X$ in $\mathcal{F}(\widehat{\mathbb{C}_c})$:

1. *i is a left map for the mapping-cocylinder CWFS*
2. *i is (the section part of) a strong deformation retract*

TFAE for $f : A \to B$ in $\mathcal{F}(\widehat{\mathbb{C}_c})$:

1. *f is a right map for the mapping-cocylinder CWFS*
2. *f is a uniform normal Kan fibration*
3. *f has uniform normal path lifting (1-box filling)*

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & A \arrow[d, "f"] \\ X \times I \arrow[r] & B \end{tikzcd}$$

Σ-*Types and* Π-*Types*

- Σ types are easy
- Π-types are more subtle, so far we only know how to get them using connections (using ideas of Gambino-Sattler and Frumin-vdBerg)

Trivial fibrations and cofibrations

Definition

- $f : A \to B$ is a **homotopy equivalence**, if there exists $g : B \to A$ and homotopies $gf \sim \mathrm{id}$ and $fg \sim \mathrm{id}$
- f is a **trivial fibration**, if it is a (normal, uniform) fibration and a homotopy equivalence
- i is a **cofibration**, if it has the llp wrt all trivial fibrations

Theorem

TFAE:

- f *is a trivial fibration*
- f *is the retract part of a strong deformation retract*
- f *admits uniform, normal right liftings wrt* $\partial I^n \hookrightarrow I^n$

TFAE:

- i *is a cofibration*
- i *is monic and has rlp wrt* $\delta : I \to I \times I$

There is a trivial-fibration/cofibration factorization (see related work by Bourke-Garner, Frumin-van den Berg, Coquand)

Constructing the impredicative universe

In general :

- $\mathcal{C}$ small category, κ inaccessible cardinal
- $F : A \to B$ in $\widehat{\mathcal{C}}$ called κ**-small**, if $\overline{F} \in \widehat{\int B}$ has κ-small fibers
- **generic** κ-small map $p : \tilde{\mathcal{U}} \to \mathcal{U}$ given by

$$\mathcal{U}(C) = \mathrm{hom}(yC, \mathcal{U}) \cong [(\int yC)^{\mathrm{op}}, \mathbf{Set}_\kappa] \cong [(\mathcal{C}/C)^{\mathrm{op}}, \mathbf{Set}_\kappa]$$

$$\tilde{\mathcal{U}}(C) = \coprod_{F \in \mathcal{U}(C)} F(\mathrm{id})$$

- perform construction in cubical sets internal to $\mathcal{E}\!f\!f$, with $\mathcal{M}$ for κ
- use arguments of Gambino-Sattler (after Cisinski) to show fibrancy and univalence
- work in progress, connections probably required

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Thanks for your attention!