Towards a realizability model of homotopy type theory

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joint work (in progress) with Steve Awodey and Pieter Hofstra

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Overview

- **Motivation**: construct *realizability model* of *homotopy type* theory, to show consistency of *impredicative univalent universe*
- Approach: internalize cubical set model in Hyland's effective topos Eff
- Context: build on related work by Coquand et al., Orton/Pitts, Gambino/Sattler, Frumin/van den Berg, Rosolini

Homotopy type theory:

Re-reading of Martin-Löf's dependent type theory where

- types are spaces
- equalities are paths

... more precisely:

Dependent Type Theory

Dependent type theory comprises:

- simple types 1, X, Y, $A \times B$, $A \Rightarrow B$, A + B, ...
- dependent types / type families $x : A \vdash B(x)$
- dependent sum types $\Sigma x : A \cdot B(x)$ and product types $\Pi x : A \cdot B(x)$
- inductive types N, list(A), ...
- identity types $x: A, y: A \vdash Id_A(x, y)$
- universes
 [™] which are 'types of types', closed under the preceding type constructors

Identity types

In the set-theoretic model, identity types are given by

$$\mathsf{Id}_{A}(x,y) = \begin{cases} \{*\} & \text{if } x = y \\ \varnothing & \text{else} \end{cases}.$$

In **locally cartesian closed categories**, identity types are modeled by diagonals

$$A \rightarrow A \times A$$
.

Interpretation satisfies uniqueness of identity proofs

not provable in type theory (Hofmann-Streicher 1994).

Irritating to classical mathematicians, but leaves room for a **homotopical interpretation**.

Identity types as path objects

Awodey-Warren (2009): interpret Id-types by **fibrant replacement** of diagonal, i.e. second part of a trivial-cofibration/fibration factorization



w.r.t. a **weak factorization system** / **WFS** (possibly part of a model structure).

Intuition: elements of $Id_A(a, b)$ are **paths** from a to b.

Lifting property of WFS corresponds to elimination rule of Id-types.

Coherence problem solved by using 'categories-with-families' and cloven WFS.

h-levels and equivalences

Types satisfying UIP can be recovered as **0-types** in HoTT.

More generally, *n*-types for $n \ge -2$ are inductively defined as follows:

• A is a (-2)-type (or contractible type), if

$$\Sigma x : A . \Pi y : A . \operatorname{Id}_A(x, y)$$

is inhabited.

• A is a (n+1)-type, if $Id_A(x,y)$ is an n-type for all x,y:A.

We call (-1)-types **propositions**, and 0-types **sets**.

Equivalences

A function $f: A \rightarrow B$ is called an **equivalence**, if its **fibers**

$$\Sigma x : A . \operatorname{Id}_B(fx, y)$$

are contractible for all y:B. equiv(A,B) is the type of equivalences from A to B.

Universes and univalence

When should two types be considered equal?

Voevodsky's **univalence axiom** asserts that two types are equal iff they are homotopy equivalent.

More precisely, a universe \mathcal{U} is called **univalent**, if the canonical map

$$\operatorname{Id}_{\mathcal{U}}(A,B) \to \operatorname{equiv}(A,B)$$

is an equivalence for all $A, B: \mathcal{U}$.

Univalence is inconsistent with UIP as soon as a type in $\ensuremath{\mathcal{U}}$ has a non-trivial automorphism.

Since classical logic implies "proof-irrelevance", it is inconsistent with univalence.

A model of HoTT with univalent universe in simplicial sets has been descibed by Voevodsky, written down by Kapulkin-Lumsdaine 2012.

Predicative and impredicative universes

 Ordinary predicative universes are closed under small products of small types:

$$\frac{\Gamma \vdash A : \mathcal{U} \qquad \Gamma, x : A \vdash B(x) : \mathcal{U}}{\Gamma \vdash \Pi x : A \cdot B(x) : \mathcal{U}}$$

 Impredicative universes are closed under arbitrary products of small types:

$$\frac{\Gamma, x : A \vdash B(x) : \mathcal{U}}{\Gamma \vdash \Pi x : A \cdot B(x) : \mathcal{U}}$$

- Subobject classifier Ω of a topos models impredicative universe of propositions.
- Impredicative universe \mathcal{U} containing a type $A : \mathcal{U}$ with two distinct elements $x \neq y : A$ inconsistent with classical logic.

Impredicative universes in realizability toposes

The **effective topos** \mathcal{E} ff (Hyland 1980) models an impredicative universe \mathcal{M} containing non-propositional types.

M is not univalent (since in topos-models, all types are 0-types)

To get an univalent, impredicative universe, need something like

- homotopical realizability model or
- realizability-∞-topos

Constructing the model internally to Eff

Observation: Existence of univalent universe in simplicial set model relies on assumption of Grothendieck universe in meta-theory.

Idea : perform model construction internally to $\mathcal{E}\!f\!f$ (containing impredicative universe) to obtain **univalent** impredicative universe.

Working internally to *Eff* imposes restrictions:

- constructive internal logic (no excluded middle)
- no transfinite constructions (no 'small object argument')

Coquand et al observed that the simplicial model relies on classical logic, proposed to use **cubical sets** instead.

Cubical sets

Cubical sets are presheaves on a cube category

- Monoidal cube category C_m used by Serre, Kan in 50ies
- Symmetric cube category C_s: free symmetric monoidal category on an interval (Bezem, Coquand, Huber 2013)
- Cartesian cube category C_c: free finite-product category on an interval / Lawvere theory with two constants
- Cartesian cube category with connections C_{cc}: Lawvere theory of distributive lattices / full subcat of Cat on objects 2ⁿ
- Lawvere theory of de Morgan algebras C_{dm} (Cohen, Coquand, Huber, Mörtberg 2016)

Comparison:

- all locally finite & can be internalized in Eff
- use C_c or C_{cc}
- \mathbb{C}_{c} much simpler than \mathbb{C}_{cc} :

$$\#\mathbb{C}_{c}([9],[1]) = 11$$
 $\#\mathbb{C}_{cc}([9],[1]) = ?$ (9th Dedekind number)

More on cube categories:

"Varieties of cubical sets" – Buchholtz, Morehouse 2017

(Iterated) path spaces in cartesian cubical sets $\mathbb C$

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[0], [1], [2],... objects of cube category.

Interval: I = Y([1])

n-cube: I^n = Y([n]) = Y([1])^n

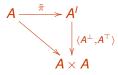
Path object: PA = A^l = A(-\times[1])

Iterated path object: P^nA = A^{l^n} = A(-\times[n])

(I tiny object, A \mapsto A^l has right adjoint – 'fractional exponent')
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Path space factorization

Awodey 2016 : algebraic weak factorization system (AWFS) on $\widehat{\mathbb{C}_{\mathsf{c}}}$ such that



is an (L, R)-factorization.

Construction uses small objects argument

To avoid this and be able to internalize in $\mathcal{E}ff$, restrict to **Kan complexes**.

Uniform normal Kan complexes

Box inclusions analogous to simplicial Horn inclusions :

$$\bigsqcup_{j}^{n} \hookrightarrow I^{n}$$
 $n \in \mathbb{N}, j \in \{\bot, \top\}$

Uniform Kan complexes have coherently chosen box fillings :

$$X \times \bigsqcup_{j}^{n} \xrightarrow{f} A$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad$$

Normality condition: fillers of 'degenerate boxes' are degenerate

$$\bigcup_{j}^{n} \longrightarrow A$$

$$\downarrow^{n-1}$$

 $\mathcal{F}(\widehat{\mathbb{C}_c}) \subseteq \widehat{\mathbb{C}_c}$ category of uniform normal Kan complexes

Cloven weak factorization systems

A cloven weak factorization system / CWFS (van den Berg, Garner 2010) on ${\cal C}$ is a functorial factorization

$$\begin{array}{cccc}
A & \xrightarrow{h} & B & & & A & \xrightarrow{h} & B \\
\downarrow f & & \downarrow g & & \mapsto & & P(f) & P(h,k) & \downarrow Lg \\
\downarrow V & & \downarrow V & & P(g) & & \downarrow P(g) \\
X & \xrightarrow{k} & Y & & X & & \downarrow P(g)
\end{array}$$

with specified fillers for all $f: A \rightarrow B$ (no naturality requirement):

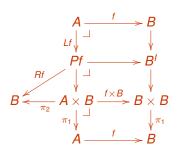
$$\begin{array}{ccc}
A & \xrightarrow{LLf} P(Lf) & Pf & \xrightarrow{id} Pf \\
Lf \downarrow & & & \downarrow RLf & & LRf \downarrow & & & \downarrow Rf \\
Pf & \xrightarrow{id} Pf & & & & P(Rf) & \xrightarrow{RRf} B
\end{array}$$

Theorem.

CWFS with stable functorial choice of diagonal factorization gives rise to model of Id-types.

A cloven CWFS on $\mathcal{F}(\widehat{\mathbb{C}_c})$

The mapping-cocylinder factorization on uniform Kan complexes gives a cloven CWFS satisfying the conditions of the theorem :



The induced WFS

Every CWFS induces a WFS with left maps *L*-coalgebras and right maps *R*-algebras.

Theorem

TFAE for $i: U \to X$ in $\mathcal{F}(\widehat{\mathbb{C}}_c)$:

- i is a left map for the mapping-cocylinder CWFS
- i is (the section part of) a strong deformation retract

TFAE for $f: A \to B$ in $\mathcal{F}(\widehat{\mathbb{C}_c})$:

- f is a right map for the mapping-cocylinder CWFS
- a f is a uniform normal Kan fibration
- f has uniform normal path lifting (1-box filling)



Σ -Types and Π -Types

- ∑ types are easy
- П-types are more subtle, so far we only know how to get them using connections (using ideas of Gambino-Sattler and Frumin-vdBerg)

Trivial fibrations and cofibrations

Definition

- $f: A \to B$ is a **homotopy equivalence**, if there exists $g: B \to A$ and homotopies $gf \sim id$ and $fg \sim id$
- *f* is a **trivial fibration**, if it is a (normal, uniform) fibration and a homotopy equivalence
- i is a **cofibration**, if it has the llp wrt all trivial fibrations

Theorem

TFAE:

- f is a trivial fibration
- f is the retract part of a strong deformation retract
- f admits uniform, normal right liftings wrt $\partial I^n \hookrightarrow I^n$

TFAE:

- i is a cofibration
- *i* is monic and has rlp wrt $\delta: I \to I \times I$

There is a trivial-fibration/cofibration factorization (see related work by Bourke-Garner, Frumin-van den Berg, Coquand)

Constructing the impredicative universe

In general:

- \mathcal{C} small category, κ inaccessible cardinal
- $F: A \to B$ in $\widehat{\mathcal{C}}$ called κ -small, if $\overline{F} \in \widehat{\int B}$ has κ -small fibers
- **generic** κ -small map $p : \tilde{\mathcal{U}} \to \mathcal{U}$ given by

$$\mathcal{U}(C) = \mathsf{hom}(yC, \mathcal{U}) \cong [(\int yC)^{\mathsf{op}}, \mathsf{Set}_{\kappa}] \cong [(\mathcal{C}/C)^{\mathsf{op}}, \mathsf{Set}_{\kappa}]$$
 $\tilde{\mathcal{U}}(C) = \coprod_{F \in \mathcal{U}(C)} F(\mathrm{id})$

- perform construction in cubical sets internal to $\mathcal{E}ff$, with \mathcal{M} for κ
- use arguments of Gambino-Sattler (after Cisinski) to show fibrancy and univalence
- work in progress, connections probably required

Bibliography, related work

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Thanks for your attention!